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In this article we construct a model in which a consumer's utility depends on the consumption history. We describe a general equilibrium framework similar to Cox, Ingersoll, and Ross (1985a). A simple example is then solved in closed form in this general equilibrium setting to rationalize the observed stickiness of the consumption series relative to the fluctuations in stock market wealth. The sample paths of consumption generated from this model imply lower variability in consumption growth rates compared to those generated by models with separable utility functions. We then present a partial equilibrium model similar to Merton (1969, 1971) and extend Merton's results on optimal consumption and portfolio rules to accommodate nonseparability in preferences. Asset pricing implications of our framework are briefly explored.

The idea that a given bundle of consumption goods will provide the same level of satisfaction at any date regardless of one's past consumption experience is implicit in models that use time-separable utility functions to represent preferences. Separable utility functions have been the mainstay in much of the literature on asset pricing and optimal consumption and portfolio

The results reported in this article were first presented at the EFA meetings in Bern, Switzerland, in 1985 [see Sundaresan (1984)]. Subsequently the article was presented at a number of universities and conferences. I thank the participants at those presentations for their feedback. I am especially thankful to Doug Breeden, Michael Brennan, John Cox, Chi-fu Huang, and Krishna Ramaswamy for their thoughtful comments and criticisms. I also thank Tong-sheng Sun for explaining the simulation procedure for stochastic differential equations and for his comments and suggestions. I am responsible for any remaining errors. Correspondence should be sent to Suresh M. Sundaresan, Graduate School of Business, Columbia University, New York, NY. Address reprint requests to Dr. Sundaresan, Graduate School of Business, Columbia University, New York, NY 10027.

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choice. The papers of Hakansson (1970), Samuelson (1969), Merton (1969, 1971, 1973), Breeden (1979), and Cox and Huang (1987) are some important examples of such contributions.[1]

Although papers using separable utility functions have provided us with many important insights, dissatisfaction with the assumption of separability is evident in the economics literature. The importance of modeling nonseparability was stressed by Fisher (1930). Criticisms of the separability assumption may also be found in Hicks (1965). In the literature on habit formation, Pollack (1970) uses utility functions that are not separable. Ryder and Heal (1973) examine the implications of such utility functions in the context of optimal growth under certainty. The assumptions about choice that give rise to nonseparable structures may be traced back to the work of Koopmans (1960) and Koopmans, Diamond, and Williamson (1964). A number of papers recently have contributed much to our understanding of how nonseparability in preferences may affect optimal behavior and market equilibrium. Some notable examples are Bergman (1985), Becker, Boyd, and Sung (1987), and Chang (1987). Also in a recent paper, Huang and Kreps (1987) discuss the role of time complementarity of consumption.[2]

In particular, time-separable utility functions imply that the marginal rates of substitution between two dates t and s ($s > t$) depend only on the consumption levels at t and s. This implication has correspondingly circumscribed the richness of the predictions of marginal utility-based asset pricing models which use separable utility functions. The recent spurt of empirical research based on nonseparable utility functions is at least in part motivated by the inability of marginal utility-based models (which use separable utility functions) to adequately explain the structure of security returns.[3] Nor are these models based on time-separable utility able to explain the remarkably stable behavior of the per capita consumption series, despite the tremendous volatility of the wealth series as proxied by the stock market wealth.

We address the problem posed by the relative stability of the consumption series by focusing on the implications of *endogenous* consumption smoothing for the equilibrium wealth process. We assume that the investor's utility from a given consumption bundle depends on the consumption history in the following way: The utility at t from a consumption level of c_t depends not only on the consumption level at t but also on the *history of consumption up to t.* This is represented by the variable z_t that is simply

[1] The contribution of Cox and Huang (1987) is of special interest as they provide a new framework for computing optimal policies and discuss the relationship between their approach and the traditional dynamic programming approach.

[2] No attempt is made to discuss the subject matter of each of these papers. Rather, our aim is to alert the reader to some of the important recent works which share a similar approach.

[3] The recent papers by Dunn and Singleton (1983, 1984) are prototypical of such empirical work.

the weighted average of past consumption rates. This variable is defined more precisely later in the article.

With this feature, we explore the optimal consumption and portfolio rules in both general equilibrium and partial equilibrium settings. The model's implications for consumption smoothing and wealth variability are investigated. We provide explicit examples of equilibrium models in which the endogenously determined consumption is much smoother than that yielded by models with separable utility. The key intuition behind most of our results is the following: An increase in consumption in response to an increase in wealth has two effects. First, it increases utility, holding the consumption standard, z_t, fixed. Second, it raises the consumption standard, z_t, which *may decrease utility.* This possibility arises by virtue of the assumptions in the specification of the intertemporal dependence in utility.[4] Thus the response to an increase in wealth is a more moderate increase in consumption. By the same token, the response to a decrease in wealth will also be shown to be more moderate in our framework. Thus, the marginal propensity to consume is generally lower in the class of models which we present in this article. This implies that any "shock" in the system must have a *relatively* greater impact on the dynamics of wealth than it would have in a model with separable utility. We find that this is in fact the case. *The ratio of the variability of changes in consumption to the variability of changes in wealth is shown to be strictly smaller in this case than in the time-separable utility case.* Consumption smoothing also occurs with time-additive utility functions. But with the type of intertemporal utility functions used in this article, we get more consumption smoothing and consequently the relative variability of consumption is much lower. The simulations presented confirm this result.

The article is organized as follows. In the next section, we present a general equilibrium framework similar to Cox, Ingersoll, and Ross (1985a). In Section 3, we solve a specific example in that general equilibrium setting. It will be shown there that the optimal consumption will involve more "smoothing" than would be implied by an additive utility function with no intertemporal dependence. The consumption volatility implications are also examined using a simulation procedure. In Section 4, we present simple partial equilibrium extensions of the important work of Merton (1969, 1971). Section 5 explores briefly the asset pricing implications of our framework. The final section concludes.

1. A General Equilibrium Framework

1.1 Assumptions

We consider in this section a single-agent economy with frictionless markets and no taxes. The assumption of a single-agent economy is standard

[4] See Assumption 2 on page 76.

and is made in the spirit of Lucas (1978) and Cox, Ingersoll, and Ross (1985a). This assumption rules out any predictions about the volume of trade in security markets. The model is one of *autarky*. The assumption of perfect markets is made so as to rule out explanations of "smoothing" which may be driven by taxes and transactions costs.

The consumer's momentary utility function, $u(c_t, z_t)$, is dependent on not only the current consumption rate c_t at time t, but also on the weighted average of the past consumption rates, z_t, where

$$z_t = z_0 + \int_0^t e^{\beta(s-t)} c_s \, ds \tag{1}$$

In Equation (1), $\beta > 0$ is a smoothing constant and c_s is the flow rate of consumption at s. The larger β is, the less weight is given to past consumption in determining z_t. By differentiating Equation (1), we obtain the following relationship:

$$dz_t = \beta(c_t - z_t)\, dt \tag{2}$$

It is worth noting from Equation (2) that the evolution of z_t is *locally nonstochastic* and depends only on the pair $\{c_t, z_t\}$. For any $s > t$, z_s will be stochastic and this will affect the stochastic properties of c_s conditional on the relevant information set at t.

The following assumptions are made concerning $u(\cdot, \cdot)$:

1. $u_c(c_t, z_t) > 0$. An increase in current consumption with no change in past consumption will increase utility.
2. $u_z(c_t, z_t) \leq 0$. An increase in past consumption with no change in current consumption will not increase current utility and may cause it to fall.
3. $u_c(c_t, z_t) + u_z(c_t, z_t) \geq 0$. An increase in a uniformly maintained consumption level will not decrease utility and may cause it to increase.
4. $u_{cc}(c_t, z_t) < 0$; $u_{cc}(c_t, z_t)u_{cz}(c_t, z_t) - [u_{cz}(c_t, z_t)]^2 \geq 0$. The utility function is strictly concave in c_t and concave in c_t and z_t.

We further assume that riskless lending and borrowing at a rate r is permitted in an instantaneously riskless market. In addition, the opportunity set is assumed to consist of many risky assets. The rate of return of asset j is governed by the following stochastic differential equation:

$$\frac{dq_t^j}{q_t^j} = \alpha_j \, dt + \sigma_j \, dB_j(t) \tag{3}$$

In the equation above, $\{B_j(t),\ t > 0\}$ is a standard Wiener process, and α_j and σ_j^2 are positive scalars. *The opportunity set is thus assumed to be stationary.*[5] The assumption about the utility function is the distinguishing feature of our model. The marginal rates of substitution between any two dates t and s depend on the entire *consumption history* up to s. As a result,

[5] Θ is the variance covariance matrix and σ_{ij} represent its elements.

we may expect the marginal utility-based asset pricing implications of our framework to be correspondingly richer.

1.2 Consumer optimization problem

The state of the economy in our setting is described by the pair $\{W_t, z_t\}$, where W_t is the wealth of the consumer. The objective of the consumer is to maximize the expected lifetime utility:

$$J(W_0, z_0) = E_0\left\{\int_0^\infty e^{-\delta t}u(c_t, z_t)\right\} \tag{4}$$

where $E_0\{\cdot\}$ is the expectations operator and δ is the subjective discount rate.

The consumer must decide, at each instant, the optimal consumption rate c_t and the optimal investment level q_t. The budget dynamics faced by the consumer are given below:

$$dW_t = \left[\sum_{j=1}^{j=N} q_t^j(\alpha_j - r) - c_t + rW_t\right]dt + \sum_{j=1}^{j=N} q_t^j\sigma_j\, dB_j(t) \tag{5}$$

Following Cox, Ingersoll, and Ross (1985a), the optimization problem may be written as shown below:

$$\max_{c_t, q_t^j}\left[u(c_t, z_t) - \delta J + J_W\left(\sum_{j=1}^{j=N} q_t^j(\alpha_j - r) - c_t + rW_t\right) + J_z\beta(c_t - z_t) + \frac{1}{2}J_{WW}\sum_{i=1}^{i=N}\sum_{j=1}^{j=N} q_t^i q_t^j\sigma_{ij}\right]$$

The existence of an interior optimum is assumed. We now proceed to derive the necessary conditions for the optimization problem. In what follows we are also not explicitly accounting for nonnegativity restrictions on consumption and investment.[6] The first-order conditions to the consumer optimization problem are stated below.

The optimality condition with respect to the consumption choice is

$$u_c(c_t, z_t) = J_W - \beta J_z \tag{6}$$

The optimality condition with respect to the investment choice is

$$J_W(\alpha_j - r) + J_{WW}\sum_{i=1}^{i=N} q_t^i\sigma_{ij} = 0,\ \forall_j \tag{7}$$

The optimality condition with respect to consumption captures the effect of nonseparability succinctly. Unlike the more traditional models in which the marginal utility of consumption is equated to the marginal utility of wealth, in our model the marginal utility of consumption is equated to the

[6] The nonnegativity constraint on investment is not binding if it can be shown that wealth is bounded away from zero. This is because the investment is equal to the wealth at each instant, at equilibrium. The equilibrium conditions are described in Section 2.3.

marginal utility of wealth *minus* β times the marginal utility of z_t. This illustrates clearly that the variable z_t will play a role in the consumption function of the agent.

1.3 Equilibrium

Given the price function $r(W_t, z_t)$ and the state dynamics, the representative individual determines the optimal consumption and investment policies as shown in the previous section. The conditions for market equilibrium are stated next. All wealth is invested in the risky production technology:

$$W_t = \sum_{j=1}^{j=N} q_t^j \tag{8}$$

The agent has *rational expectations*. The price function and the state dynamics assumed by the agent in solving the optimization problem are the *actual* price functions and state dynamics which are implied by the agent's decisions. This way of closing the model is standard and follows from Lucas (1978).

1.4 General results

Result 1.[7] The equilibrium interest rate is given by the expected rate of change in the marginal utility of wealth.

This result is simply theorem 1 of Cox, Ingersoll, and Ross (1985a) extended to our setting. Formally, this result is stated below:[8]

$$r = \delta - \frac{L\{J_W\}}{J_W} \tag{9}$$

Later, by imposing additional restrictions, we explicitly solve for the equilibrium interest rate.

Result 2. The price of a default-free discount bond, $P(t, s)$, at t which pays 1 unit of the consumption good at date s is given by the expected marginal rate of substitution between t and s.

This result has been derived by Rubinstein (1974) in the context of separable utility functions. Formally this result is stated below:

$$P(t, s) = E_t\left\{\frac{J_{W_s}(W_s, z_s)}{J_{W_t}(W_t, z_t)}\right\} \tag{10}$$

While the functional forms are identical to the ones obtained by Cox, Ingersoll, and Ross (1985b), it is useful to note that the value functions now depend on the variable z_t and significant economic differences can arise as a result of this.

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[7] We have omitted proofs of Result 1 and Result 2 for brevity.

[8] The notation L is used to describe the differential generator.

It is worth noting that the wealth process at equilibrium may also be interpreted as the asset price process.[9]

2. A General Equilibrium Example

The general framework presented in Section 2 is now specialized by imposing more structure on preferences and the opportunity set as shown below.

- The utility function will be specialized as follows:

$$u(c_t, z_t) = -\frac{1}{\phi_1} e^{-\phi_1 c_t + \phi_2 z_t} \tag{11}$$

In the equation above, $\phi_1 > \phi_2 \geq 0$. The parameter ϕ_2 determines the strength of intertemporal dependence.

- The opportunity set specified in Equation (3) will be further simplified as follows: The opportunity set will now be assumed to consist of a number of instantaneous production technologies whose returns are *independently and identically distributed* and have constant returns to scale with a mean rate of return α and variance σ^2. Let χ be the minimum input necessary to produce a positive amount. Given a wealth of q_t, the consumer will *optimally* invest an equal amount in each technology. This follows from Samuelson (1969). Thus the problem may be seen as one in which the agent invests q_t in q_t/χ technologies. Then the evolution of wealth may be *approximated* as

$$dq_t = q_t \alpha dt + \hat{\sigma}\sqrt{q_t}\, dB \tag{12}$$

where $\hat{\sigma}$ is a scalar equal to $\sigma\sqrt{\chi}$.

This approximation provides a useful rationalization for the wealth dynamics that are shown in the equation above. Cox and Ross (1976) point out this interpretation. Alternatively, it may be assumed directly that the output from the technology follows the dynamics above, although this is less intuitive since the technology is not stochastic constant returns to scale.

The case considered in this section corresponds to the intertemporally dependent utility function case with constant absolute risk aversion. We report below the key results and discuss their principal implications.[10] To solve for the value function, consumption rule, and interest rate, we substitute Equations (11) and (12) into the first-order condition (6), (7), and the Bellman equation specified in Section 2.2. The resulting Bellman equation is a partial differential equation in W_t and z_t. This is solved to obtain the value function which is of the form:

[9] This follows from the fact that the market price of an asset that has a claim to the output from a risky technology i is simply the stock of the good in that technology. We, however, focus on the consumption smoothing implications of intertemporally dependent utility functions.

[10] The proof is tedious and offers little insight. Hence it is omitted.

$$J(q_t, z_t) = -b_1 e^{-b_2 q_t + b_3 z_t} \tag{13}$$

Next, the value function is used in Equations (6) and (7) to obtain the consumption function and the interest rate.

The optimal consumption policy is linear in wealth and the variable z_t.

$$c_t = \frac{b_2}{\phi_1} q_t + \frac{\phi_2 - b_3}{\phi_1} z_t - \frac{1}{\phi_1} \ln \frac{b_1(b_2 + \beta b_3)}{\phi_1} \tag{14}$$

The equilibrium interest rate is a constant and is given by the expression:

$$r = \alpha - \frac{b_2}{2}\hat{\sigma}^2 \tag{15}$$

The constants of the value functions may be solved and found to be given by the expressions below:

$$b_3 = \frac{-\theta + \sqrt{\theta^2 + 8\alpha\beta\phi_2\hat{\sigma}^2}}{2\beta\hat{\sigma}^2} > 0$$

where

$$\theta = \beta(\phi_1 - \phi_2)\hat{\sigma}^2 + 2(\beta + \alpha) > 0$$

$$b_2 = \frac{2\phi_1\alpha - 2\beta b_3}{\hat{\sigma}^2\phi_1 + 2}$$

$$b_1 = \frac{\phi_1}{b_2 + \beta b_3} e^{\left[1 - \frac{\delta\phi_1}{(b_2 + \beta b_3)}\right]} \tag{16}$$

It is useful to note that $b_i > 0$, for any i and $\theta > 0$.

In this model, the ratio of the variability of consumption to the variability of wealth is

$$\frac{\sigma^2(dc_t)}{\sigma^2(dq_t)} = \frac{b_2^2}{\phi_1^2} \tag{17}$$

To sharply contrast these results with the time-separable case, it is useful to first restate the corresponding results for the base case when $\phi_2 = 0$, which serves as a convenient *benchmark* since it implies no intertemporal dependence in preferences. Setting $\phi_2 = 0$ and $\beta = 0$ in Equation (13), it may be shown that the value function is

$$J(q_t) = -\frac{1}{a}\{e^{(\alpha - \delta\phi_1)/a - aq_t}\} \tag{18}$$

where[11] $a \equiv 2\alpha\phi_1/(2 + \phi_1\hat{\sigma}^2)$.

The constant parameter a may be interpreted as the measure of constant

[11] See Sundaresan (1983) for details.

absolute risk aversion of the indirect value function. The optimal consumption policy is

$$c_t = \frac{aq_t}{\phi_1} + \left(\frac{\delta}{a} - \frac{1}{\phi_1}\right) \tag{19}$$

Note from Equation (19) that the model also implies the following relationship between the volatility of changes in consumption and the volatility of changes in wealth:

$$\frac{\sigma^2(dc_t)}{\sigma^2(dq_t)} = \frac{a^2}{\phi_1^2} > 1 \tag{20}$$

Comparing Equation (17) with Equation (20), and noting that $b_2 < a$, we can draw the following important conclusion: *The ratio of the variance of consumption changes to the variance of wealth changes with intertemporal dependence is strictly less than the corresponding one for the base case when* $\phi_2 = 0$. This implication is stated next:

$$\frac{\sigma^2(dc_t)}{\sigma^2(dq_t)} = \frac{b_2^2}{\phi_1^2} < \frac{a^2}{\phi_1^2} < 1 \tag{21}$$

This illustrates our key finding that nonseparable utility functions lead to greater consumption smoothing behavior.

To assess the extent of consumption smoothing, it is useful to compare the behavior of *growth rates of consumption* in the base case and in the case corresponding to $\phi_2 > 0$. Since the consumption growth rates depend on the dynamic properties of q_t and z_t, it becomes necessary to either explicitly solve for the conditional density of $\{q_t, z_t\}$ or to simulate the paths of these variables. Since the conditional density of q_t was difficult to obtain in closed form even in the base case, it became necessary to use the simulation approach which is explained in detail next.

The key to the simulation procedure is the discrete time approximation for the equilibrium wealth process.

The wealth dynamics in the base case follow the process:

$$dq_t = \kappa\{\mu - q_t\}\,dt + \hat{\sigma}\sqrt{q_t}\,dB(t)$$

where

$$\kappa \equiv \frac{-\alpha\phi_1\hat{\sigma}^2}{2 + \phi_1\hat{\sigma}^2} > 0$$

and

$$\mu \equiv \frac{\alpha - \phi_1\delta}{\kappa\alpha\phi_1}$$

The classification of boundaries for this process has been done by Feller

(1952). The boundaries for this process can be *regular or entrance or absorbing* depending on the values of the parameters.[12]

The simulation procedure was based on the discrete-time analog similar to the one suggested in Sun (1987). The dynamics of q_t may be written in a *discrete-time* analog as follows:

$$q_{n,n+k} - q_{n,n} = \kappa\mu k - \kappa \sum_{i=0}^{k-1} q_{n,n+i} + \hat{\sigma} \sum_{i=0}^{k-1} q_{n,n}^{1/2} \epsilon_{n+i+1}, \qquad n = 0, 1, 2, \ldots$$

where $\epsilon_1, \epsilon_2, \ldots$ are identically and independently distributed normal random variables with zero mean and unit variance. This discrete-time analog was used where a period of one year was divided into 3000 subintervals to get the desired accuracy. The subscript n refers to time period in years, and each year is divided into k subperiods for the purposes of sample path simulation.[13]

In selecting the parameters for the simulation, the empirical findings of Mehra and Prescott (1985) were used as a broad guideline. Mehra and Prescott (1985) document that during 1889–1978, the mean real riskless rate was about 0.80%. The instrument used in their study to compute the riskless rate was 90 day Treasury bills. They also document that the real return on S&P 500 during 1889–1978 averaged about 6.98%.

For the simulations, we employed a regular boundary which ensures that the consumption good stock will always be nonnegative, given a positive initial value.

The simulation was carried out for a period of 30 years and the results are shown in Table 1 for three cases. The sample mean and the standard deviation of the consumption series were computed for the consumption series. (All annualized.)

Case B presented below sets $\phi_2 = 2$ and case C which is presented next sets $\phi_2 = 3$. In both cases, β is varied from 1 to 5. The results are tabulated below.

It is clear from the table that as beta increases, the variance of the consumption growth rate increases as well: intuitively, an increase in beta causes less weight to be assigned to past consumption and the variability approaches that generated by an intertemporally *independent* utility function. The effect depends on the value of ϕ_2. This is also evident from the table. As ϕ_2 increases, the strength of historical dependence increases. At all levels of β, the variability of the growth rate of consumption is lower than the corresponding figures reported for case A in the table.

The volatility of consumption changes in these cases is less than the base case although this has not been reported in the table.

[12] See Feller (1952) for the restrictions which lead to different boundary classifications. When $\kappa \times \mu > \hat{\sigma}^2/2$, then the boundary is entrance: This implies that zero is inaccessible. When $0 < \kappa \times \mu < \hat{\sigma}^2/2$, then the boundary is regular: This implies that zero is accessible. But the consumption good stock will always be nonnegative. When $\kappa \times \mu < 0$, the boundary is absorbing and the consumption good stock will converge to zero with probability 1.

[13] Each simulation run took about 12 minutes of computer time.

Table 1
Mean and standard deviation of consumption growth rates

Regime	Mean growth rate, %	Standard deviation of growth rate, %
A. Base case		
$\phi_2 = 0$	11.66	26.18
B. $\phi_2 = 2$		
$\beta = 1$	5.75	5.95
$\beta = 2$	5.88	8.11
$\beta = 5$	6.08	11.26
C. $\phi_2 = 3$		
$\beta = 1$	7.05	6.62
$\beta = 2$	7.40	10.62
$\beta = 5$	8.09	18.61

All results assume expected return on production technology, $\alpha = 6.98$ percent; coefficient of risk aversion, $\phi_1 = 4$; the production technology diffusion coefficient, $\hat{\sigma} = 1.965$; subjective discount factor $\delta = 0.70$ percent. The parameters in the simulation procedure were calibrated to yield a riskless rate and risk premium that are broadly consistent with the evidence reported above. The parameter values chosen implied that $\kappa \times \mu > 0$, which is required to maintain a nonnegative consumption good stock.

The evidence presented here is only indicative of the possibility that nonseparable utility functions may be able to reconcile the observed stickiness in the consumption series. We have provided a reasonable theoretical explanation of this important regularity. Only by empirical tests of this issue with nonseparable utility functions can one completely resolve this regularity.

3. A Partial Equilibrium Example

In this section, we extend the partial equilibrium model of Merton (1969, 1971) to our framework. Unlike our earlier discussions, we will now assume that the wealth process is exogenously specified as shown in Equation (3) and that the consumer may borrow or lend at an *exogenously determined constant riskless rate r,* and investigate the consumer's optimal investment and consumption strategies. A direct extension of Merton's CPRA utility function is presented next.

This[14] example considers the following utility specification:

$$u(c_t, z_t) = \frac{\{c_t - z_t\}^A}{A} \tag{22}$$

where $A < 1$.

This utility function has the property that as $c_t \to z_t$, the marginal utility tends to ∞. So the level z_t serves as a natural "floor level of consumption" below which the consumer will never allow the consumption rate to fall. We will show that *this floor is endogenous and increases with time and that it is increasing in wealth.*

[14] CPRA stands for Constant Proportional Risk Aversion. Note that in this example, we apply this concept with respect to the argument, $c_t - z_t$, inside the utility function.

Merton (1969, 1971) shows that, *without the z_t variable,* the optimal consumption policy and investment policies are as shown below.[15]

$$c_t = \lambda W_t \tag{23}$$

where

$$\lambda = \frac{\delta - rA - \tfrac{1}{2}[A/(1-A)][(\alpha - r)^2/\sigma^2]}{1 - A}$$

$$q_t = \frac{(\alpha - r)W_t}{(1-A)\sigma^2} \tag{24}$$

These results are obtained by Merton by first solving a finite horizon problem and then taking the limit to the infinite horizon case. All our solutions that follow in this section were obtained in the same way. The optimal solution will preclude wealth from becoming negative and hence the individual always consumes a positive amount.

The value function, optimal consumption policy, and investment policy are provided below for the utility function specified in (22):

$$J(W_t, z_t) = \Omega\{rW_t - z_t\} \tag{25}$$

where

$$\Omega = \left(\frac{1}{r+\beta}\right)\left\{\frac{1-A}{\delta - rA - \tfrac{1}{2}[A/(1-A)][(\alpha - r)^2/\sigma^2]}\right\}^{1-A}$$

$$c_t = z_t + \{rW_t - z_t\}\Psi \tag{26}$$

where[16]

$$\Psi = \frac{\delta - rA - \tfrac{1}{2}[A/(1-A)][(\alpha - r)^2/\sigma^2]}{(1-A)(r+\beta)}$$

$$q_t = \frac{[rW_t - z_t][\alpha - r]}{(1-A)r\sigma^2} \tag{27}$$

It is useful to note that, if the initial endowment W_0 is such that $rW_0 > z_0$, then $rW_t - z_t$ is lognormally distributed with a state space of $(0, \infty)$. This ensures that $rW_t > z_t$, and hence the value function derived above makes economic sense.

The following properties are worthy of mention: In the absence of intertemporal dependence, as Merton has shown, the optimal policy is to consume a constant fraction of wealth. The prediction of our model is that the optimal consumption is a convex combination of the riskless income on current wealth, rW_t, and the past consumption standard, z_t. Clearly, this

[15] Cox and Huang (1987) extend Merton's result to take into account nonnegativity restrictions on consumption and wealth.

[16] We assume that $\Psi > 0$, so as to ensure that $c_t > z_t$. The proof of this result is omitted for brevity. Interested readers may prove this result by following the steps in Merton (1971).

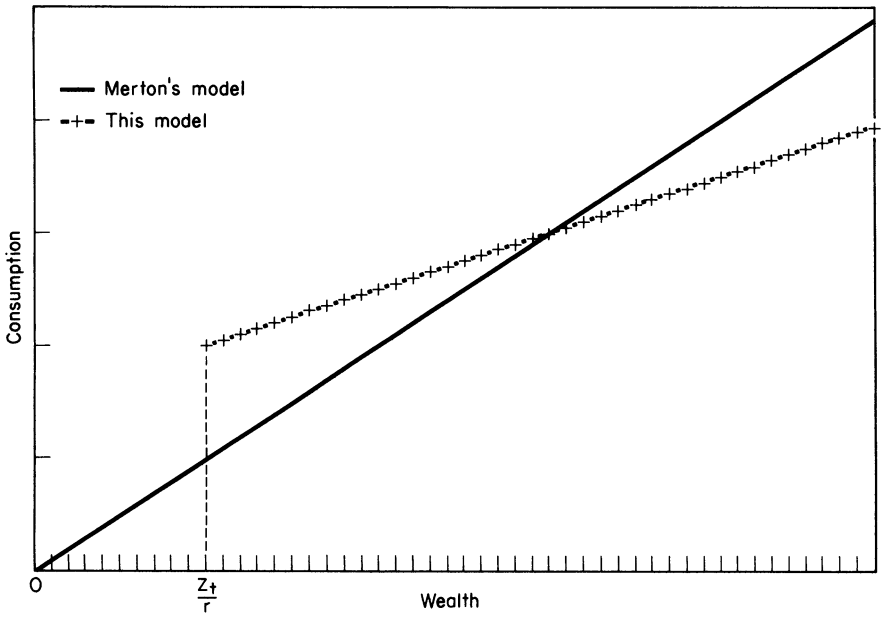

Figure 1
Optimal consumption policies (impact of nonseparability)
The line passing through the origin represents the optimal consumption function derived by Merton. The line with a *smaller slope* beginning at z_t/r is the optimal consumption function corresponding to this model. The variable z_t is the past consumption standard and r is the constant riskless rate of interest.

implies significant smoothing of consumption. The marginal propensity to consume out of wealth is strictly lower with intertemporal dependence so long as β is greater than zero. In Figure 1, the optimal consumption policy is compared with that derived by Merton (1969, 1971). Furthermore, the optimal policy is always to consume an amount greater than z_t which is increasing over time:

$$\frac{dz_t}{dt} = \beta\{c_t - z_t\} > 0$$

The optimal consumption rate tends to increase with time, holding wealth fixed. In addition, z_t tends to be higher as wealth level increases. This observation follows from the fact that the optimal consumption policy is increasing in wealth. Since z_t is an integral of the consumption history, in periods of increasing wealth, we expect it to increase.

The fraction of wealth invested in the risky asset is *no longer a constant,* but an increasing concave function of wealth. Only when the wealth approaches ∞ does the proportion of wealth invested asymptotically approach the policy found by Merton (1969, 1971). The optimal investment policy is to invest *a constant proportion of the wealth in excess of the capitalized value of the consumption standard, $W_t - z_t/r$, in the risky asset. In the portfolio insurance literature, this is sometimes referred to as investing a constant proportion of the "cushion" in the risky asset.* See

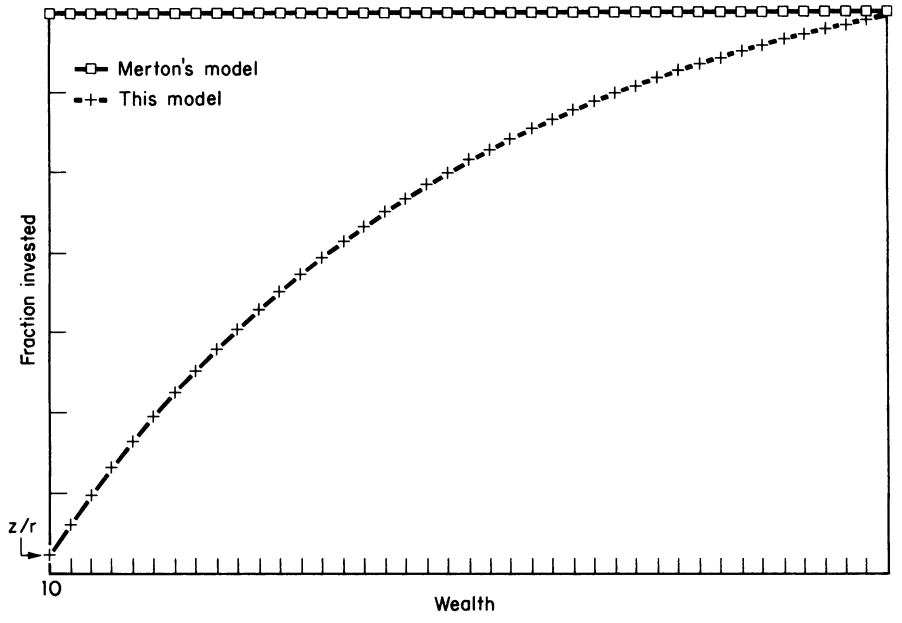

Figure 2
Optimal investment policies (impact of nonseparability)
The horizontal line at the top represents the fraction invested in the riskless asset in the model due to Merton. The curve represents the fraction invested in the risky asset according to this model. Note that the initial wealth level is set at z_t/r, where r is the riskless rate of interest and z_t is the past consumption standard.

Black and Perold (1987). The optimal investment policies are distinguished from the corresponding ones derived by Merton (1969, 1971) in Figure 2.

4. Asset Pricing Implications

The asset pricing implications of nonseparable utility functions were discussed in some detail by Bergman (1985). We offer in this section some observations based on our earlier results which complement his results.

When the opportunity set is nonstochastic, a single β consumption-based CAPM will hold even when the utility functions are represented by the specification discussed in Section 2.1. The intuition for this result should be direct: The only source of randomness is the wealth process and, at equilibrium, wealth changes and consumption changes are locally perfectly correlated. *This partial equilibrium result extends the consumption-based CAPM to the class of nonseparable utility functions used in this paper.* The Sharpe CAPM and the multi-β CAPM derived by Merton also are valid with this utility specification.

When the opportunity set is stochastic, Cox, Ingersoll, and Ross (1985a)

have shown that the asset returns may be represented in the following manner:

$$\mu_s - r = -\frac{1}{J_W} cov\left(\frac{ds}{s}, dJ_W\right)$$

In the equation above, μ_s is the vector of expected returns and s is the random instantaneous asset returns. It is possible to reduce this relationship to the following multifactor relationship.[17]

$$\mu_s - r = -\frac{u_{cc}}{J_W} V_{sc} + \left(-\frac{J_{zW}\beta}{J_W}\right) V_{sW} + \sum_{i=1}^{N} \left(-\frac{J_{zx_i}\beta}{J_W}\right) V_{sx_i}$$

In the multifactor CAPM shown above, V_{sx_i} are the covariances of returns of assets with the state variable i. *Note that despite the consumption β which is the first term on the right-hand side, additional factors become necessary to explain the structure of expected excess returns.*

These results are to be contrasted with those of Merton (1973), Breeden (1979), and Bergman (1985). The asset pricing structure has a multi-β representation similar to Merton (1973), while the single-consumption β representation obtained by Breeden (1979) no longer obtains in this setting. These conclusions reinforce the results of Bergman (1985).[18] The results also suggest that factors other than aggregate consumption may be relevant in describing the cross-sectional variations in security returns.

5. Conclusions

This article provides a simple family of nonseparable utility functions to explain the observed consumption smoothing. In addition, we have shown that the ratio of volatility in consumption changes to the volatility in wealth changes is much less in this family of models than in comparable models with separable utility functions. Since it was not possible to derive the conditional density of the consumption function in closed form, we simulated the sample path of consumption to get some insights into its behavior over time. It was found that our general equilibrium model with nonseparable utility functions was capable of generating more sticky consumption than the general equilibrium model where the utility function was separable. The impact of nonnegativity constraints on consumption and investment has not been treated in this work. The results of Cox and Huang (1987) may be applied in the context of nonseparable utility functions used in this article. In addition, the endogenous *stochastic* evo-

[17] This may be done by expanding the change in the marginal utility of wealth by applying *Ito's* lemma.

[18] In his paper, Bergman (1985) showed that the single-consumption β CAPM may not hold when the utility function is nonseparable.

lution of the term structure is yet another area of interest. These are topics of current research.

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