The Distribution of Stock Returns

R. R. OFFICER*

A detailed examination is made of the distribution of stock returns following reports that the distribution is best described by the symmetric stable class of distributions. The distributions are shown to be "fat-tailed" relative to the normal distribution but a number of properties inconsistent with the stable hypothesis are noted. In particular, the standard deviation appears to be a well behaved measure of scale.

1. INTRODUCTION

An appropriate method of characterizing and summarizing the behavior of a random variable is to describe it in terms of a distribution function. There is no natural law that determines which particular function accurately describes the distribution of the variable, if indeed any analytic function accurately describes it. In practice, the properties of the sampling distribution are compared with the properties of distribution functions so that a suitable one can be selected. The purpose of this article is to describe the distribution of stock returns.

One of the findings of the study is that the distribution of stock returns has some characteristics of a non-normal generating process. In particular, in line with other studies [3, 1, 11] the results indicate the distribution is "fat-tailed" relative to a normal distribution. However, characteristics were also observed which are inconsistent with a *stable* non-normal generating process. Evidence is presented illustrating a tendency for longitudinal sums of daily stock returns to become "thinner-tailed" for larger sums, but not to the extent that a normal distribution approximates the distribution. Further, the standard deviation as a measure of scale appears to be well behaved.

2. PREVIOUS EVIDENCE ON THE DISTRIBUTION OF STOCK RETURNS

Mandelbrot[1] was mainly responsible for the reexamination of the distribution of stock returns in the context of non-normal stable distributions. Previous work[2] had concluded that the normal distribution was a good working hypothesis. Fama [3] made the first detailed study of stock returns in the context of stable distributions. He concluded that the distribution of monthly returns belonged to a non-normal member of the stable class of distributions. Blume [1] examined the distribution of monthly residuals estimated from the market model;[3] his results were consistent with Fama's. More recently Teichmoeller [11] examined the distribution for daily returns and sums up to 10 days. He concluded the distribution belonged to a non-normal member of the stable class, but it had somewhat "fatter tails" (smaller characteristic exponent) than that found by Fama and Blume.

3. SYMMETRIC STABLE DISTRIBUTIONS

Since the studies of Fama and Blume, the properties and the estimation of the parameters of stable distributions have been examined in detail by Fama and Roll [4, 5]. The stable class of distributions and their important properties are discussed in their papers together with a complete bibliography so that a detailed discussion of the distributions will be bypassed here.

The most distinguishing feature of symmetric non-normal stable distributions is peakedness and fat tails when compared with the normal distribution. The parameter of stable distributions which measures the degree of peakedness and the fatness of the tails is the characteristic exponent (α). The range of α that this study is concerned with is bounded by the normal distribution ($\alpha=2$) and the Cauchy distribution ($\alpha=1$). These two distributions are the only distributions in this range for which a probability density function is known in closed form. The density function of other distributions can be estimated using Bergstrom's series expansion.[4]

The procedure used to obtain estimates of α is described by Fama and Roll [5]. This procedure makes use of the property of a monotonic decline in the values of higher fractiles, e.g., .95, of symmetric stable distributions as α increases. The Studentized Range (SR) was also estimated, where SR = range/standard deviation. The SR was found by Fama and Roll to be the best of a number of goodness-of-fit tests of normality against non-normal stable alternatives.

Initially, this article reports the results of an examina-

* R.R. Officer is senior lecturer, Department of Business Administration, University of Queensland, St. Lucia, Queensland, Australia 4067. The author has benefitted from helpful comments and criticism of this article by R. Blattberg, E.F. Fama, N. Gonedes, and H.V. Roberts. The research was carried out while the author was at the University of Chicago.

[1] For example, see [8].

[2] Most of these studies are reproduced in [2].

[3] The market model is defined as

$$R_{jt}=\gamma_j+\beta_j R_{mt}+\epsilon_{jt},$$

where R_{jt} is the rate of return on stock j at time t; R_{mt} is the market factor, usually an index of stock returns; γ_j and β, are parameters specific to stock j and ϵ_{jt} is the disturbance term.

[4] This expansion series is described in [4].

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Applications Section

tion of the stationarity of the mean estimate of ($\hat{\hat{\alpha}}$), and therefore the constancy of α of the distribution of stock returns through time. Next the stability of the distributions as reflected by $\hat{\alpha}$ is examined longitudinally (sums of stock returns through time) and cross-sectionally (sums of returns across stocks, i.e., portfolio returns). This examination proceeds under the assumption that the distribution of stock returns is a member of the stable class of distributions. However, even if this assumption is not true the estimates of $\hat{\alpha}$ are still a measure of the "fatness of the tails" for the type of sample distributions under consideration here, i.e., reasonably symmetric and well behaved. A lower $\hat{\alpha}$ of the distribution of returns for one period compared with another period indicates a greater number of relative outliers in the period of lower $\hat{\alpha}$.

Following the examination of $\hat{\alpha}$, other properties of the distributions are examined (Sections 5 and 6) such as stability and the behavior of the scale parameter. These properties are pertinent to the question of whether we can approximate the distribution of stock returns by a member of the stable class of distributions.

4. STATIONARITY AND THE DISTRIBUTION OF STOCK RETURNS

4.1 The Distribution of Monthly Returns

In the first test $\hat{\alpha}$ was estimated for the distribution of monthly returns of a random sample of 39 stocks, listed continuously from January 1926 to June 1968, i.e., 509 observations. The period is the full time covered by the CRSP price relative tape.[5] The results are given in Table 1(a) and clearly indicate a non-normal distribution with an $\hat{\alpha}$ of about 1.51.

For the second test the period was divided into two equal subperiods of 254 observations. Period 1 was from February 1926 to March 1947 and Period 2 was from May 1947 to June 1968. Estimates of α of the distribution of monthly returns were made for Period 1 on 78 stocks and Period 2 on 136 stocks. These stocks were listed continuously over their respective periods and they reflect the different proportions of stocks listed over these periods on the CRSP tape. The 39 stocks examined in the first test were also examined for the two periods; they are also included in the samples of Period 1 and 2 stocks. The results are given in Table 1, parts (b) and (c).

The results suggest different distributions for Periods 1 and 2. However, the results are not conclusive since stock returns are not independent. The market factor relates the price movements of stocks to each other, explaining up to 50 percent of the variation in price relatives for some periods, see [7]. Moreover, it is possible that the market factor could be generated by a process with a constant α and for the range of $\hat{\alpha}$ for the market factor to vary between the ranges of $\hat{\hat{\alpha}}$ given by the Period 1 and 2 data. For example, Fama and Roll's [5] results for 299 simulations of a stable distribution with $\alpha = 1.7$ and a sample size of 199 gave a 0.1 fractile of the distribution of the estimates of $\hat{\alpha}$ as 1.54 and the 0.9 fractile as 1.84.

1. CHARACTERISTICS OF THE DISTRIBUTION OF MONTHLY STOCK RETURNS

	Characteristic exponent ($\hat{\hat{\alpha}}$)[a]		Studentized range[b] ($\hat{SR}$)	
(a) Sample 39 stocks. Period 2/1926-6/1968 (509 observations)				
	1.51 (0.12)[c]		10.72 (1.70)	
	Period 1(2/1926-3/1947, n = 254)		Period 2 (5/1947-6/1968, n = 254)	
	$\hat{\hat{\alpha}}$	$\hat{SR}$	$\hat{\hat{\alpha}}$	$\hat{SR}$
(b) Sample 39 stocks				
	1.49 (.13)	8.46 (1.23)	1.74 (.17)	6.70 (.74)
(c)	Returns - sample 78 stocks		Returns - sample 136 stocks	
	1.48 (.11)	8.75 (1.45)	1.75 (.18)	6.74 (1.22)
(d)	Residuals - sample 78 stocks		Residuals - sample 136 stocks	
	1.52 (.112)	8.972 (1.777)	1.71 (.289)	6.847 (1.649)
(e) Market factor				
	1.40	8.45	1.74	6.39

[a] $\hat{\hat{\alpha}}$ indicates an average of the estimates of α, i.e., for (a) $\hat{\hat{\alpha}} = 1/39 \sum_{j=1}^{39} \hat{\alpha}_j$.

[b] Percentage points of the distribution of SR from a normal population were taken from David, Hartley, and Pearson, "The Distribution of the Ratio in a Single Normal Sample of Range to Standard Deviation," *Biometrica*, 41 (1954), 482–93:

N	*Lower 5 percent*	*Upper 5 percent*
200	4.78	6.38
500	5.36	6.94

[c] The figures in parentheses are standard deviations. The distribution of $\hat{\alpha}$ and SR were reasonably symmetric and well behaved. Further, $\hat{\alpha}$ is bounded by values of 2.0 and 1.0. Under these circumstances one might expect the distributions of $\hat{\alpha}$ to be roughly approximated by a normal distribution. A comparison of the fractiles from the sample distributions and a normal distribution with identical mean and standard deviation gave roughly similar results.

To overcome the problem of dependence the residuals from the market model[6] were examined. If all stock returns are generated by the same stable process then the residuals must also belong to that distribution.[7] Moreover, evidence from King [7] and Blume [1] indicate the residuals are approximately independent. In the third test on the distribution of monthly stock returns the residuals were estimated for the two sets of stocks and the two subperiods of the previous test. The results are shown in part (d) of Table 1. A comparison of parts (c) and (d) of Table 1 indicates some narrowing of $\hat{\hat{\alpha}}$ of the distribution of residuals with respect to the returns but the results still suggest different distributions for the two subperiods.

All the results of this section indicate a change in the distribution from Period 1 to Period 2 (approximately pre- and postwar). Clearly, there may be no simple dichotomy of the distributions such as pre- and postwar. There may be a continual change in the distribution but because of the lack of observations we are in no position to test this for monthly data. Instead, we must rely on inferences drawn from an examination of daily data to decide whether it is likely that the distribution is continually changing.

[5] This tape is described by Fisher and Lorie [6].

[6] See Footnote 4.

[7] This follows from the property of stability.

4.2 The Constancy of the Characteristic Exponent for Daily Stock Returns

A random sample of 50 stocks was taken from the sample of 136 stocks examined in the postwar period. There was only one condition for selection: all stocks had to be listed over the entire period of the Scholes' Daily Stock Returns Tape,[8] i.e., 7/2/62 to 6/11/69. This period was split into eight subperiods, each with 217 observations (trading days). It was considered that this number of observations would be required to give reasonably accurate estimates of α.

The complete results are not shown, but the mean $\hat{\alpha}$, i.e., $\hat{\bar{\alpha}}$, for the fifty stocks varied between 1.61 and 1.68 for the eight subperiods and each subperiod had a standard deviation (i.e., $SD_{\hat{\alpha}}$) of approximately .15. Similar results were obtained for the residuals of the fifty stocks from the market model; the range was 1.61 to 1.67 for the eight subperiods with comparable levels of standard deviation to the above. Clearly the distribution of stock returns, judged by $\hat{\alpha}$ values, has not changed substantially over the period. The apparent stationarity of the distribution in the 1960's provides some evidence that the differences in the distributions found for Periods 1 and 2 were not likely due to continual changes in the distribution.

5. SOME PROPERTIES OF THE DISTRIBUTION OF STOCK RETURNS

5.1 Stability

The central issue at this stage is whether the distribution of stock returns behaves as though the generating process was a non-normal stable distribution; more particularly, is this a good working model? The first test is to examine whether stock returns exhibit the important property of stability, i.e., sums of independent stable variables with a characteristic exponent α have a distribution with the same characteristic exponent α. To test for stability $\hat{\alpha}$ was computed for the 39 stocks listed continuously from January, 1926 to June, 1968 for sums of monthly returns up to five months. If the stock returns were all distributed by a stable distribution whose parameters were constant over time,[9] then the results should show smaller $\hat{\alpha}$'s for larger intervals. The decrease in $\hat{\alpha}$ results from the increasing downward bias in $\hat{\alpha}$'s as the size of the sample decreases [5]. Alternatively, if the process was one suggested by Press [10], i.e., the returns are distributed as though they were drawn from normal distributions with a changing scale parameter (standard deviation), then the α's should increase with length of interval, since the Central Limit Theorem will be applicable and $\hat{\alpha}$ should approach 2.0 for larger sums.

The results, not tabulated, show no tendency for the $\hat{\alpha}$ of larger sums to change from the $\hat{\alpha}$ of the single values. Overall the results suggest that at least for sums up to five not much is lost by assuming stability for monthly returns.

The preceding was concerned with the longitudinal tests of stability of the distribution of stock returns, but there is also the problem of the property of stability cross-sectionally. Cross-sectional stability means that portfolios of stocks have the same distribution as the component stocks that make up the portfolio.[10] If stock returns were generated by a normal process but with non-constant scale parameters we might expect through the Central Limit Theorem that the $\hat{\alpha}$ of sums of stocks (portfolios) would approach 2.0 as the size of the portfolio increases. But stock returns are not independently distributed cross-sectionally. Therefore, one way to test whether the theorem holds is to examine the distribution of sums of the residuals cross-sectionally from the market model.

The results are given in Table 2 and indicate that while the portfolio $\hat{\alpha}$ are good approximations of the $\hat{\bar{\alpha}}$ of the component stocks for returns, this is not true for residuals. The results conflict with the stable hypothesis, i.e., that sums of independently distributed random variables from a stable process with characteristic exponent α sum to give a stable distribution with the same α.

2. THE RELATIONSHIP BETWEEN PORTFOLIO AND STOCKS[a]

Size of portfolio	Returns				Residuals			
	Portfolio		Stocks		Portfolio		Stocks	
	$\hat{\bar{\alpha}}$	$\hat{\overline{SR}}$	$\hat{\bar{\alpha}}$	$\hat{\overline{SR}}$	$\hat{\bar{\alpha}}$	$\hat{\overline{SR}}$	$\hat{\bar{\alpha}}$	$\hat{\overline{SR}}$
				Period 1, 2/1926 - 3/1947				
Ten stocks (n=7)[b]	1.50 (.173)	8.92 (.982)	1.48 (.045)[c]	8.71 (.523)	1.72 (.139)	9.23 (2.224)	1.52 (.038)	9.26 (.631)
Twenty stocks (n=3)	1.48 (.098)	9.35 (.548)	1.47 (.098)	8.79 (.351)	1.58 (.052)	10.33 (2.561)	1.52 (.031)	8.94 (.543)
Thirty stocks (n=2)	1.44 (.050)	9.43 (.090)	1.47 (.020)	8.79 (.349)	1.66 (.102)	10.60 (2.351)	1.52 (.000)	8.94 (.634)
				Period 2, 5/1947 - 6/1968				
Ten stocks (n=13)	1.79 (.091)	6.12 (.675)	1.74 (.052)	6.75 (.160)	1.85 (.118)	6.25 (1.160)	1.74 (.052)	6.75 (.160)
Twenty stocks (n=6)	1.85 (.042)	6.04 (.720)	1.74 (.016)	6.75 (.158)	1.88 (.087)	6.16 (.782)	1.71 (.105)	6.82 (.442)
Thirty stocks (n=4)	1.72 (.061)	5.99 (.028)	1.74 (.030)	6.75 (.090)	1.92 (.081)	5.64 (.453)	1.65 (.265)	6.82 (.399)

[a] The $\hat{\bar{\alpha}}$ of stocks is the average characteristic exponent of stocks making up the portfolio.

[b] n is the number of portfolios.

[c] The standard deviation of the components' $\hat{\bar{\alpha}}$ is the standard deviation of the mean $\hat{\alpha}$ of the component stocks for each portfolio.

5.2 The Behavior of the Scale Parameter

It was shown that the distribution of stock returns over time appears to be reasonably stable for sums up to five. Thus an additional check on the appropriateness of assuming stability for small sums is to examine the behavior of the parameters of the distribution.

An important parameter from the point of view of portfolio theory is the scale parameter (c) which measures the degree of dispersion of the distribution. If it can be shown that the scale parameter of the distribution behaves in a

[8] This tape was made available by Wells Fargo Bank, San Francisco.

[9] Although evidence was presented that the parameters of the monthly returns were not constant the property of stability may still be reasonably well approximated. We have no theoretical way of predicting how stable distributions will behave with respect to this property for changing α.

[10] Although it is not reported here, some evidence was found that the distribution of returns of major stocks may have a larger α than returns of stocks in general. In a pilot study of 30 major stocks for the postwar period, it was difficult to reject the hypothesis of normality for the distribution of monthly returns.

predictable fashion for sums of stock returns, then any measure of risk that is a function of the scale parameter is independent of the time interval, e.g., daily, monthly, yearly, etc. Any ranking of portfolios on the basis of the scale parameter will be constant, irrespective of the time interval over which the estimates of the parameter were obtained.

Fama and Roll [4] have shown that an estimate of c can be obtained from the sample fractiles and this estimate is almost independent of the particular stable distribution for $1.0<\alpha<2.0$. The fractile range used is the .28 fractile to the .72 fractile. Just as the standard deviation has a value of unity for a standardized normal distribution, by construction $\hat{c}=(\hat{X}_{.72}-\hat{X}_{.28})/2(.827)$ has approximately the same value for standardized stable distributions. Thus $\hat{c}$ can be used in a similar manner for stable distributions as the sample standard deviation is used for the normal distribution to describe dispersion.

The standard deviation of linear combinations (l.c.) of independent normal variates is

$$\sigma \text{ of l.c.} = \left[\sum_{j=1}^{n} a_j^2 \sigma_j^2 \right]^{\frac{1}{2}}.$$

The scale parameter c of linear combinations of independent variates drawn from the same stable distribution is

$$c \text{ of l.c.} = \left[\sum_{j=1}^{n} | a_j c_j |^{\alpha} \right]^{1/\alpha}.$$

Thus the stability of a random variable can be tested by examining the $\hat{c}$ of non-overlapping sums of stock returns, where the sum is a linear combination with $a_j=1/n$. If the random variable is stable and independent,[11] we should find that

$$\hat{c}(\text{sum}) = \left[\sum_{j=1}^{n} \hat{c}_j^{\alpha} \right]^{1/\alpha} = (n\hat{c}^{\alpha})^{1/\alpha},$$
$$= n^{1/\alpha}\hat{c}$$

where n is the number in the sum. Similarly, for the standard deviation,

$$\hat{\sigma}(\text{sum}) = \left[\sum_{j=1}^{n} \hat{\sigma}_j^2 \right]^{\frac{1}{2}} = (n\hat{\sigma}^2)^{\frac{1}{2}},$$
$$= n^{\frac{1}{2}}\hat{\sigma}.$$

Table 3 shows the results of testing these two alternative measures of the scale parameter. The same stocks (39) for the same period (January 1926–June 1968) were used in the test as the tests of the longitudinal stability of $\hat{\alpha}$. The differences between the actual estimates of the scale parameter of the sum and those calculated from the single values, using the formulas given above, are shown as percentages of this single value estimate.

Overall, the results are consistent with the hypothesis that the scale parameter is invariant for sums of stock returns to five months. Only one of the estimates is more than two standard deviation units away from zero and even for that estimate (s.d. for a sum of three) it is only just over. The results indicate that the c for an α of between 1.7 and 1.8 exhibited greatest consistency of the estimates of c. The $\hat{\alpha}$ of the same group of stocks was

[11] Given the wealth of evidence in support of the Random Walk Theory of stock returns [2] independence is a reasonable assumption.

3. TESTS OF STABILITY OF ALTERNATIVE MEASURES OF SCALE FOR MONTHLY DATA[a]

Scale parameter	Mean	Average absolute percentage difference[b] sums of monthly returns				Average percentage difference sums of monthly returns			
		Two	Three	Four	Five	Two	Three	Four	Five
c(1.5)	12.623	7.949 (5.265)	12.801 (8.170)	14.398 (8.625)	15.346 (10.105)	7.322	12.606	12.429	14.870
c(1.6)	9.695	6.105 (4.826)	9.651 (7.583)	11.315 (8.153)	11.707 (8.689)	4.606	8.512	7.222	8.965
c(1.7)	8.204	5.152 (4.396)	8.050 (6.473)	9.893 (8.072)	9.721 (8.091)	2.144	4.741	2.371	3.416
c(1.8)	8.334	5.186 (4.024)	7.619 (5.809)	9.896 (8.902)	10.634 (7.377)	-.098	1.259	-2.154	-1.800
c(1.9)	9.391	5.615 (4.238)	8.028 (5.966)	11.334 (9.985)	12.588 (8.195)	-2.147	-1.964	-6.380	-6.705
s.d.	6.203	3.176 (2.384)	9.596 (4.402)	6.177 (5.458)	5.862 (4.319)	-.165	9.596	-2.196	.292

[a] The period covered was from 2/26 to 6/68.

[b] These figures are the percentage difference from the predicted value of the sum. They are calculated as $[(\hat{c}(\text{sum}(n)) - (n\cdot\hat{c}_1^{\alpha})^{1/\alpha})/(n\cdot\hat{c}_1^{\alpha})^{1/\alpha}]\cdot 100$, where α covers the range 1.5 to 2.0, and $\hat{c}_1$ is the estimate of the scale parameter for individual observations.

about 1.6 corrected for bias, so there is some inconsistency between these two methods of estimating $\hat{\alpha}$. Estimating $\hat{\alpha}$ from the $\hat{c}$ estimates is comparable to the range analysis method used by Fama [3].

A perhaps surprising result of this test is that the standard deviation (SD) is well behaved. If we were dealing with a true stable process with $\alpha\neq 2.0$, we would not expect the SD to exhibit the same invariance as the interfractile range (IFR). As a further test on the behavior of the standard deviation as a measure of scale, the same tests were performed on daily returns for sums up to 20 trading days, i.e., up to a month over the period 7/2/62–6/11/69. Because daily tests are conducted over a much shorter time span they are less likely than the monthly tests to run into the problem of non-constancy of the parameters of the distribution, which could invalidate the previous test. The results are reported in the next section.

5.3 Stability of Distribution of Daily Stock Returns

The same type of test for stability of the distribution as described in Section 4.1 was performed on daily stock returns. The same sample of 50 stocks used in the test of time series changes in the characteristic exponent described in Section 3.2 was used in this part of the study. Thirty-six of the stocks in this sample were in the sample of 39 stocks used in the test of stability of the characteristic exponent of the distribution for monthly returns (Section 5.1). The characteristic exponent was estimated for daily returns and sums of 2, 3, 4, 5, 7, 10, 15 and 20 daily returns. There were 1,738 observations for daily returns down to 85 observations for the sum of 20 daily returns. The results of the analysis are summarized in Table 4.

4. THE CHARACTERISTIC EXPONENT FOR SUMS OF DAILY RETURNS[a]

Sums of daily returns								
1	2	3	4	5	7	10	15	20
Characteristic exponent								
1.50 (.090)	1.59 (.094)	1.62 (.110)	1.63 (.113)	1.68 (.137)	1.71 (.138)	1.72 (.148)	1.75 (.185)	1.73 (.153)

[a] Over the period 7/2/62–6/11/69 for a sample of 50 stocks.

The results in Table 4 show a tendency for the $\hat{\alpha}$ to increase for larger sums, whereas we would expect the reverse because of the downward bias in the estimates of $\hat{\alpha}$ as the number of observations decline [5]. The tendency for $\hat{\alpha}$ to increase for larger sums is slight, nonetheless, the fact that the $\hat{\alpha}$ of sums of daily stock returns do increase[12] suggests a modified model with a finite second moment for the distributions.[13] The Central Limit Theorem gives no indication of the rate normality will be approached for drawings from distributions with finite second moments.

The behavior of alternative measures of scale for the distributions of sums of daily stock returns was also tested. The method of testing was the same as the tests for the monthly data described in Section 4.1 and 4.2. The results are given in Table 5, which is comparable to Table 3 for the monthly data. Once again the standard deviation appears to be a good measure of scale. On the basis of Table 5 it appears that the standard deviation is superior to the other measures of scale, although this type of test is sensitive to any serial correlation in the sample.[14]

5. *TESTS OF SCALE MEASURES FOR DAILY DATA*[a]

Scale measure	Sums of daily returns							
	Two	Three	Four	Five	Seven	Ten	Fifteen	Twenty
				Average absolute percentage differences				
c(1.2)	18.006 (7.781)	27.137 (7.080)	33.081 (7.843)	36.933 (7.634)	42.449 (8.437)	48.028 (7.097)	53.917 (7.437)	59.023 (7.041)
c(1.3)	14.418 (7.882)	21.821 (7.597)	26.901 (8.445)	30.079 (8.464)	34.803 (9.557)	39.762 (8.225)	45.181 (8.847)	50.348 (8.532)
c(1.4)	11.408 (7.811)	16.956 (8.069)	21.432 (8.367)	23.614 (9.247)	27.446 (10.636)	31.637 (9.335)	36.386 (10.266)	41.464 (10.058)
c(1.5)	9.232 (7.351)	12.628 (8.301)	16.539 (8.344)	17.754 (9.571)	20.914 (10.703)	23.715 (10.416)	27.630 (11.679)	32.489 (11.601)
c(1.6)	7.760 (6.905)	9.430 (7.775)	12.603 (7.986)	13.332 (8.656)	15.393 (10.453)	16.636 (10.553)	20.323 (10.830)	23.741 (12.719)
c(1.7)	7.178 (6.371)	7.964 (6.555)	9.780 (7.781)	10.518 (7.592)	12.392 (9.060)	12.215 (8.900)	14.787 (9.903)	16.951 (11.826)
c(1.8)	7.088 (6.170)	7.716 (5.731)	8.895 (7.410)	9.300 (7.479)	11.550 (8.668)	10.833 (7.966)	12.722 (9.428)	13.706 (10.238)
c(1.9)	7.577 (6.015)	8.240 (5.795)	9.318 (7.842)	9.961 (8.181)	12.679 (9.623)	12.062 (9.394)	13.938 (11.322)	14.717 (9.949)
s.d.	3.008 (2.371)	3.690 (3.065)	4.883 (3.826)	6.310 (4.176)	7.164 (5.432)	8.286 (6.041)	8.848 (7.293)	10.963 (7.473)

[a] The period covered was from 7/2/62 to 6/11/69. The figures in the body of the table were estimated in the same manner as that described for Table 3.

These results do not mean that it is inappropriate to use a non-normal stable distribution to approximate the distribution of stock returns. Clearly the distributions over the time intervals studied here have "fat tails," so that the normal distribution is going to give a poor approximation of the distribution. The well behaved nature of the SD of the distributions suggests that the distributions will have some properties which true non-normal stable distributions do not have, e.g., with this property the use of least-squares estimation methods can be better justified. It may be that a class of "fat-tailed" distributions with finite second moments will be found to give better approximations of the distributions of stock returns, but as yet this remains to be clearly demonstrated.[15]

6. CONCLUSIONS

At the start of this article it was stated that no natural law predetermines stock returns to conform to any particular distribution. Following the previous work [1, 3, 11] the distribution of stock returns was examined in the context of stable distributions. The results indicate that the returns have some but not all the properties of a stable process. The distributions have "fat tails" compared to the normal distribution. Monthly stock returns behave consistently with the property of stability, at least for sums up to five months.

On the other hand there was a tendency for estimates of the characteristic exponent ($\hat{\alpha}$) of the distribution of daily returns to increase for larger sums, e.g., sums of 10, 15 and 20 daily returns. Also the standard deviations of these sums were well behaved. If the process was a true stable one with $\alpha<2.0$, then we would expect any estimate of the standard deviation to behave erratically [3, 5]. If we are concerned with the second moment of the distribution of stock returns, then these results suggest an analytic distribution function for which the second moment is finite may be a more appropriate model.

Other inconsistencies with the stable hypothesis were also observed. Cross-sectional sums of monthly stock residuals from the market model had an $\hat{\alpha}$ that was larger than the $\hat{\bar{\alpha}}$ of the components of the sum. This difference was observed for both pre- and postwar distribution.

The main implications of these findings can be summarized as follows:

1. When a stable distribution is used to characterize the distribution of monthly returns, it appears reasonable, as an approximation, to assume that the $\hat{\alpha}$ of portfolios is approximately the same as that for stocks. However, it was found that the $\hat{\alpha}$ of cross-sectional sums of residuals appear to increase relative to the $\hat{\alpha}$ of the components.

2. It is appropriate to assume a stable distribution of monthly stock returns with $\hat{\alpha}\simeq 1.8$ postwar and 1.5 prewar when tail areas are under examination.

3. Monthly returns to stocks appear to be reasonably stable, at least for sums up to five months. This property does not hold as well for sums of daily returns up to 20 days, although the distributions of the sums are not wildly erratic judging by the behavior of the $\hat{\alpha}$ (Table 4) and measures of dispersion (Table 5).

4. The sample standard deviation appears to be a well behaved measure of dispersion.[16]

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[12] The results are consistent with Teichmoeller's [11] findings, but he does not draw attention to the increase in $\hat{\alpha}$ for larger sums and concludes his results are consistent with the stable hypothesis.

[13] The sums of logs of returns to the market index gave identical results as arithmetic sums. It is unlikely that rates of returns up to 20 days are going to be large enough ($\simeq$15 percent) for sums of logs to differ substantially from single sums.

[14] The same types of tests were run on the data used by Fama [3], i.e., the 30 stocks of the Dow-Jones Industrial Index covering the period approximately 1958–1962. The pattern of results was entirely consistent with the findings presented here. Also tests were made comparing the SD and the mean absolute deviation (MAD), which in contrast to the SD has a finite first moment for non-normal members of stable distributions, for variability as moving series. The SD was slightly less variable than the MAD.

[15] Work by Praetz [9] on Australian stock market data suggests a t-distribution with varying degrees of freedom. As yet unpublished work by Blattberg and Gonedes at the University of Chicago suggests t-distributions also show promise for U.S. stock market data.

[16] A comparison of the standard deviation, the mean absolute deviation and the fractile range used by Fama and Roll using a simple linear regression model showed them all to be closely related for the distribution of stock returns (r^2's were around 0.9).

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