Financial markets as adaptive ecosystems

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Option markets offer an interesting example of adaptation of a population (the traders) to a complex environment, through trial and error and natural selection (unefficent traders disappear quickly). The problem is the following: an 'option' is an insurance contract protecting its owner against the rise (or fall) of financial assets, such as stocks, currencies, etc. The problem of knowing the value of such contracts became extremely acute when organized option markets opened twenty five years ago, allowing one to buy or sell options much like stocks. Almost simultaneously, Black and Scholes (BS) proposed their famous option pricing theory, based on a simplified model for stock fluctuations, namely the (geometrical) Brownian motion model. The most important parameter of the model is the 'volatility' σ, which is the standard deviation of the market price's relative fluctuations. Guided by the Black-Scholes theory, but constrained by the fact that 'bad' prices lead to arbitrage opportunities, option markets agree on prices which are close, but significantly and systematically different from the BS formula. Surprisingly, a detailed study of the observed market prices clearly shows that, despite the lack of an appropriate model, traders have empirically adapted to incorporate some subtle information on the real statistics of price changes.

More precisely, a 'call' option is such that if the price $x(T)$ of a given asset at time T (the 'maturity') exceeds a certain level x_c (the 'strike' price), the owner of the option receives the difference $x(T) - x_c$. Conversely, if $x(T) < x_c$, the contract is lost. To make a long story short [1, 2, 3], if T is small enough (a few months) so that interest rate and average returns are negligible compared to fluctuations, the 'fair' price $\mathcal{C}$ of the option today ($T = 0$), knowing that the price of the asset now is x_0 is given by:

$$\mathcal{C}(x_0, x_c, T) = \int_{x_c}^{\infty} dx' \; (x' - x_c) P(x', T|x_0, 0) \tag{1}$$

where $P(x', T|x_0, 0)$ is the conditional probability density that the stock price at time T will be equal to x', knowing its present value is x_0. Eq. (1) means that the option price is such that on average, there is no winning party. Pricing correctly an option is thus tantamount to having a good model for the conditional probability $P(x', T|x_0, 0)$.

There is fairly strong evidence that beyond a time scale τ of the order of ten minutes, the fluctuations of stock values (on the major markets) are uncorrelated [4], but *not identically distributed* variables [5]. More precisely, one can write:

$$x(T) = x_0 + \sum_{k=0}^{\frac{T}{\tau}-1} \delta x_k \tag{2}$$

where the increments δx_k are distributed as:

$$P(\delta x_k) \equiv \frac{1}{\gamma_k} P_0 \left(\frac{\delta x_k}{\gamma_k} \right) \tag{3}$$

which means that δx_k is obtained as the product of a random variable, the distribution of which is independent of k, times a *scale factor* γ_k which stochastically depends on k (see below).

Let us first consider the case where $\gamma_k = \gamma_0$ independent of k, corresponding to the classical problem of a sum of independent, identically distributed variables. Although $P(\delta x)$ is strongly non Gaussian (see, e.g. [6]), the Central Limit Theorem [7] tells us that for $N = T/\tau$ large, $P(x', T|x_0, 0)$ will be close to a Gaussian. Using then Eq. (1) essentially leads back to the BS formula (although in principle BS use a log-normal, rather than a normal distribution, the difference is not relevant for the present discussion). For finite N, however, there are corrections to the Gaussian, and thus corrections to the BS price. A useful way to characterize these corrections is to introduce the *cumulants* of the elementary increments δx. To a very good approximation, the distribution $P_0(\delta x)$ is even [6]; a classical result is then that the leading correction when N is large is proportional to the *kurtosis* κ, defined as $\kappa = \langle \delta x^4 \rangle / \langle \delta x^2 \rangle^2 - 3$ [7], which vanishes if δx is itself Gaussian, and measures the 'fatness' of the tails of the distribution as compared to a Gaussian. It is then easy to show that the leading correction to the BS price can be reproduced by using the BS formula, but with a modified value for the volatility $\sigma = \sqrt{\langle \delta x^2 \rangle}$ (which traders call the 'implied volatility' Σ), which depends both on the strike price x_c and on the maturity T through:

$$\Sigma(x_c, T = N\tau) = \sigma \left[1 + \frac{\kappa}{24N} \left(\frac{(x_c - x_0)^2}{\sigma^2 N} - 1 \right) \right] \tag{4}$$

which is called the 'smile effect', because the plot of Σ versus x_c has the shape of a smile (see Fig 1). That the volatility had to be smiled up was realized long ago by traders – this reflects the well known fact that the elementary increments have rather 'fat' tails: markets are much more jerky than what a Gaussian random walk would look like.

As shown in Fig. 1, the smile formula (4) reproduces correctly the observed option prices on the 'Bund' market *provided the kurtosis* κ becomes itself N dependent. The shape of the 'implied' kurtosis $\kappa_{\text{imp}}(N)$ as a function of N is given in Fig. 2; $\kappa_{\text{imp}}(N)$ is seen to increase steadily. Why is this so?

Let us study directly the kurtosis of the distribution of the underlying stock, $P(x, T|x_0, 0)$, as a function of $N \equiv T/\tau$. If the increments δx were independent and identically distributed (i.e. $\gamma_k \equiv \gamma_0$), one should observe that $\kappa_N = \kappa/N$. In Fig. 2, we have also shown $\kappa_{\text{eff}} = N\kappa_N$ as a function of N. One can notice that not only κ_{eff} is not constant (as it should if δx were identically distributed), but actually κ_{eff} matches quantitatively (at least for $N \leq 200$) with the evolution of the implied kurtosis κ_{imp}! In other words, the price over which traders agree capture rather precisely the anomalous evolution of κ_{eff}.

As we shall show now, this non trivial behaviour of κ_{eff} is related to the fact that the *scale* of the fluctuations γ_k is actually itself a time dependent random variable [5]. This could come from the fact that new information induces reactions of arbitrary sign, increasing the scale of fluctuations; conversely, when fluctuations are too large, risk-averse operators leave the market and this decreases the scale of fluctuations. It is thus reasonnable to assume that γ_k oscillates around a mean value, with random fluctuations, possibly correlated in time. Writing $\gamma_k^2 = 1 + \eta_k$, with $\overline{\eta_k \eta_q} = g(|k-q|)$ ($\overline{(...)}$ refers now to an average over the scale fluctuations), one finds that Eq. (4) still holds, but with κ replaced by an effective kurtosis κ_{eff} given by:

$$\kappa_{\text{eff}}(N) = \kappa_0 + (\kappa_0 + 3)g(0) + 6 \sum_{j=1}^{N} (1 - \frac{j}{N}) g(j) \tag{5}$$

where κ_0 is the kurtosis of $P_0(\delta x)$. The simplest possibility is that η_k follows an Ornstein-Uhlenbeck process [7], in which case $g(k) = g_0 \alpha^k$, where $\alpha < 1$ is related to the correlation time of the η variable. The solid line in Fig. 2 shows a rather good fit of $\kappa(N)$ with this formula, with $\kappa \simeq 20$, $g_0 \simeq 4$ and $\alpha = 0.9913$, corresponding to a correlation time of 7.2 days. Note that the effect of a non zero kurtosis on the BS prices was previously investigated in [8, 9], although the relation between κ_{eff} and κ_{imp}, or their N dependence, was not investigated.

In conclusion, we have shown by studying in detail the market price of options that traders have evolved from the simple, but inadequate BS formula to an empirical know-how which encodes two important statistical features of asset fluctuations: 'fat tails' (i.e. a rather large kurtosis) and more subtle non stationary effects (i.e. the fact that the scale of fluctuations is itself time dependent). These features, although not explicitly included in the theoretical pricing models used by traders, are nevertheless reflected rather precisely in the price fixed by the market as a whole. Financial markets thus behave as adaptive systems with efficient emerging properties.

Figure Captions.

Fig 1: Example of a smile curve: Implied volatility $\Sigma(x_c, T)$ *vs* distance from strike price $(x - x_c)$ for a given T. The data shown correspond to all 227 transactions of December options on the German Bund future (LIFFE) on November 13, 1995. This is a very 'liquid' market, meaning that price anomalies are expected to be small, in particular for short maturities T. Both call and put options are included with put options transformed into call options using the put-call parity [2]. Volatilities are expressed as annualized standard deviation of price differences. According to Eq. (4) the data should fall on a parabola. From a fit of the curvature of this parabola, we extract the 'implied kurtosis' κ_{imp} for a given $N = \frac{T}{\tau}$. In this particular case we find $\kappa_{\text{imp}} = 276$ at $N = 144$ (9 trading days).

Fig 2: Plot of the implied kurtosis κ_{imp} (determined as in Fig. 1) and of the historical kurtosis κ_{eff} (determined directly from the historical movements of the Bund contract), as a function of the reduced time scale $N = \frac{T}{\tau}$, $\tau = 30$ minutes. All transactions of options on the Bund future from 1993 to 1995 were analyzed along with 5 minute tick data of the Bund future for the same period. The growth of the error bars for the latter quantity comes from the fact that less data is available for larger N, and that a factor N comes in the definition of κ_{eff}. Finally, a fit with formula (5), corresponding to a simple Ornstein-Uhlenbeck evolution of the scale parameter γ_k is shown for comparison. This allows one to extract a correlation time for these fluctuations of the order of a week.

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