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Testing for Nonlinear Dependence in Daily Foreign Exchange Rates*

The purpose of this article is to investigate whether daily changes in five major foreign exchange rates contain any nonlinearities. Although the data contain no linear correlation, evidence indicates the presence of substantial nonlinearity in a multiplicative rather than additive form. Further examination reveals that a generalized autoregressive conditional heteroskedasticity (GARCH) model can explain a large part of the nonlinearities for all five exchange rates.

I. Introduction

The purpose of this article is to investigate whether changes in foreign exchange rates exhibit nonlinear dependence. For almost an entire decade, the stylized fact about exchange rates is that they behave like random walks. Let S_t denote the U.S. dollar price of a unit of foreign currency at date t. Mussa (1979) observes that $s_t = \log(S_t)$ can be described as a random walk. Meese and Rogoff (1983) show that simple random walk models dominate structural models in terms of predictive performance in foreign exchange rates. The random walk model is characterized as follows. Let $x_t = s_t - s_{t-1}$ be the logarithmic growth rate of the exchange rate. If x_t is *statistically independent* of past observations $x_{t-1}, x_{t-2}, \ldots$, then s_t follows a random walk.

A random walk model of exchange rates has some unappealing implications. It means that the sample path and the (unconditional) variance of

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the levels of exchange rates are unbounded. It also imposes some restrictions on models of exchange rate determination, as pointed out in Manas-Anton (1986). I note further than a random walk exchange rate does not imply, nor is it implied by, rational expectations or market efficiency.[1]

Thus far, there has been no strong statistical evidence confirming the random walk model. Most investigations have focused on the linear predictability (or lack thereof) of exchange rate changes. At best, the data suggest that exchange rate changes are *uncorrelated.* This is not sufficient to prove statistical independence, in view of the nonnormality of x_t.[2] It is possible for exchange rate changes to be linearly uncorrelated and nonlinearly dependent. Theoretically, there is no reason to believe that economic systems must be intrinsically linear. Empirically, there is some evidence that exchange rate changes exhibit nonlinear dependence. Hsieh (1988*a*) rejects the null hypothesis that x_t is independent and identically distributed (iid) for daily price changes of five currencies from 1974 to 1983 and finds that this rejection can be attributed to changing means and variances. Manas-Anton (1986) shows that x_t^2 is serially correlated, even though x_t itself is not.

One explanation of the nonlinear dependence is that exchange rate changes are purely deterministic processes that "look" random. An excellent summary of chaotic systems can be found in Scheinkman and LeBaron (in this issue). This article will instead focus on a second explanation of nonlinear dependence: that exchange rate changes are nonlinear stochastic functions of their own past. I employ a method proposed by Brock, Dechert, and Scheinkman (1987) to test directly for nonlinear dependence and attempt to distinguish between different types of nonlinearity. I find substantial evidence of nonlinear dependence in daily exchange rate changes, which implies that new theories must be developed to account for this stylized fact.

II. Nonlinear Stochastic Systems

Empirical work has been leading theoretical research in nonlinear time-series models. Priestley (1980) proposes a general framework to handle nonlinear time series. This class of models is much richer than linear time-series models. Here are several well-known examples. Robinson (1979) proposes the nonlinear moving average model, the simplest one being

$$x_t = \epsilon_t + \alpha \, \epsilon_{t-1} \, \epsilon_{t-2}. \tag{2.1}$$

1. Lucas (1978) contains a general equilibrium asset-pricing model in which asset prices do not necessarily behave as random walks.
2. See Burt, Kaen, and Booth (1977), Westerfield (1977), Rogalski and Vinso (1978), Manas-Anton (1986), and Hsieh (1988*a*).

Tong and Lim (1980) deal with the threshold autoregressive model, such as

$$x_t = \alpha x_{t-1} + \epsilon_t, \quad \text{if } x_{t-1} \leq 1,$$
$$= \beta x_{t-1} + \epsilon_t, \quad \text{otherwise.} \tag{2.2}$$

Granger and Andersen (1978) introduce the bilinear time-series model, as in

$$x_t = \epsilon_t + \alpha\, x_{t-1}\, \epsilon_{t-1}. \tag{2.3}$$

In the three examples above, ϵ_t is a sequence of normal iid random variables. A fourth example of a nonlinear system is Engle's (1982) autoregressive conditional heteroskedasticity (ARCH) model, of which the simplest is

$$x_t = \epsilon_t, \tag{2.4}$$

where ϵ_t is conditionally normally distributed, with zero mean and variance $h_t = [\alpha + \phi\, x_{t-1}^2]$.

Time series generated by these systems all exhibit little or no serial correlation, and yet x_t is not stochastically independent of x_{t-1}. Thus, traditional tests of linear dependence (such as autocorrelation coefficients, run tests, etc.) will not detect the nonlinear dependence.

There has been little theoretical work in economics that gives rise to nonlinear time-series models. This is, of course, not a weakness of these nonlinear models. Rather, it is indicative of the difficulty in solving and analyzing nonlinear stochastic general equilibrium models. The nonlinear moving average and the bilinear model can be justified as second-order approximations of the Wold representation, which stationary (linear and nonlinear) time series possess. Threshold autoregressive and ARCH models have enjoyed slightly more attention in the theoretical literature. A threshold autoregressive model can be found in Aiyagari, Eckstein, and Eichenbaum (1985). The idea of their paper is that the price of a storable good will switch between two linear stochastic processes, depending on whether inventory is positive or zero. Hsieh (1988*b*) presents a nonlinear stochastic rational expectations model of exchange rates under central bank intervention. This is a nonlinear switching model, similar to a threshold autoregression.

The most popular nonlinear model in empirical econometric work has been ARCH, which is very useful in describing heteroskedasticity in many economic time series. There are, of course, theoretical reasons to believe that conditional moments are important determinants of asset prices since most intertemporal asset-pricing models give rise to Euler equations that involve conditional expectations of marginal utilities across assets and across time periods. It is therefore not hard to visualize conditional variances and covariances showing up in asset demand functions. The problem in generating an ARCH specification is that the Euler equation must be solved out completely to obtain a characterization of the asset prices, a difficult task when the model is nonlinear. Recently, Lai and Pauly (1988) offer a model in which speculators in the foreign exchange market with incomplete information about market fundamentals use a Kalman filter to extract information and form expectations. The equilibrium exchange rate then turns out to be an autoregressive moving average (ARMA) process with conditionally heteroskedastic errors.

III. Testing Nonlinearity: The BDS Statistic

A method to test for nonlinear dependence makes use of the idea of the "correlation integral." Given a time series $\{Z_t: t = 1, \ldots, T\}$ of D-dimensional vectors. Define the correlation integral $C(\ell)$ as

$$C(\ell) = \lim_{T\to\infty} \frac{2}{T(T-1)} \sum_{i<j} I_\ell(Z_i, Z_j), \tag{3.1}$$

where $I_\ell(x, y)$ is an indicator function that equals one if $\| x - y \| < \ell$, and zero otherwise, where $\| \;\; \|$ is the sup-norm. The correlation integral $C(\ell)$ measures the fraction of the pairs of points of $\{Z_t\}$ that are within a distance of ℓ from each other.

The correlation integral is used by Grassberger and Procaccia (1983) to define the "correlation dimension" of $\{Z_t\}$:

$$\nu = \lim_{\ell\to 0} [\log C(\ell)/\log \ell], \quad \text{if the limit exists.} \tag{3.2}$$

Physicists use the correlation dimension to distinguish between chaotic deterministic systems and stochastic systems. However, the lack of a proper statistical theory is a drawback to the analysis. In fact, Ramsey and Yuan (1987) show that the estimated correlation dimension may be substantially biased even in samples with as many as 2,000 observations.

This article follows an alternative strategy, proposed by Brock, Dechert, and Scheinkman (1987), referred to as the BDS test. Instead of trying to distinguish a chaotic system from a stochastic system, they propose to test the null hypothesis that the data are independently and identically distributed, using a procedure that has power against both deterministic chaos and nonlinear stochastic systems.

The BDS test computes a statistic based on the correlation integral. Let $\{x_t: t = 1, \ldots, T\}$ be a sequence of observations that are independent and identically distributed. Form N-dimensional vectors $x_t^N = (x_t, x_{t+1}, \ldots, x_{t+N-1})$. These are called "$N$-histories." Calculate the correlation integral

$$C_N(\ell, T) = \frac{2}{T_N(T_N - 1)} \sum_{t<s} I_\ell(x_t^N, x_s^N), \tag{3.3}$$

where $T_N = T - N + 1$. Brock, Dechert, and Scheinkman (1987) show that under the null hypothesis $\{x_t\}$ is iid with a nondegenerate density F,

$$C_N(\ell, T) \to C_1(\ell)^N \quad \text{with probability one, as } T \to \infty,$$

for any fixed N and ℓ. Furthermore, they show that $\sqrt{T}[C_N(\ell, T) - C_1(\ell, T)^N]$ has a normal limiting distribution with zero mean and variance:

$$\sigma_N^2(\ell) = 4\,[K^N + 2 \sum_{j=1}^{N-1} K^{N-j}C^{2j} + (N-1)^2C^{2N} - N^2KC^{2N-2}], \tag{3.4}$$

where

$$C = C(\ell) = \int [F(z + \ell) - F(z - \ell)]dF(z),$$

$$K = K(\ell) = \int\!\!\int [F(z + \ell) - F(z - \ell)]^2 dF(z).$$

Note that $C_1(\ell, T)$ is a consistent estimate of $C(\ell)$, and

$$K(\ell, T) = \frac{6}{T_N(T_N - 1)(T_N - 2)} \sum_{t<s<r} I_\ell(x_t, x_s)\, I_\ell(x_s, x_r) \tag{3.5}$$

is a consistent estimate of $K(\ell)$. Thus, $\sigma_N(\ell)$ can be estimated consistently by $\sigma_N(\ell, T)$, which uses $C_1(\ell, T)$ and $K(\ell, T)$ in place of $C(\ell)$ and $K(\ell)$. Under the null hypothesis, the BDS statistic

$$w_N(\ell, T) = \sqrt{T}\,[C_N(\ell, T) - C_1(\ell, T)^N]/\sigma_N(\ell, T) \tag{3.6}$$

has a standard normal limiting distribution.

This test has an intuitive explanation. The correlation integral $C_N(\ell, T)$ is an estimate of the probability that any two N-histories, $x_t^N = (x_t, x_{t+1}, \ldots, x_{t+N-1})$ and $x_s^N = (x_s, x_{s+1}, \ldots, x_{s+N-1})$, are within ℓ of each other, that is,

$$C_N(\ell, T) \to \text{prob}\{|x_{t+i} - x_{s+i}| < \ell \quad \text{for all } i = 0, 1, \ldots, N-1\}, \quad \text{as } T \to \infty.$$

If the x_t's are independent, then, for $|t - s| > N$,

$$C_N(\ell, T) \to \prod_{i=0}^{N-1} \text{prob}\{|x_{t+i} - x_{s+i}| < \ell\}, \quad \text{as } T \to \infty.$$

Furthermore, if the x_t's are also identically distributed, then

$$C_N(\ell, T) \to C_1(\ell)^N, \quad \text{as } T \to \infty.$$

Simulations in Brock, Dechert, and Scheinkman (1987) show that this test has power against simple nonlinear deterministic systems as well as nonlinear stochastic processes.

The BDS statistic gives some information about the type of dependence in the data. Suppose $w_N(\ell, T)$ is a positive number. The probability of any two N-histories, $(x_t, x_{t+1}, \ldots, x_{t+N-1})$ and $(x_s, x_{s+1}, \ldots, x_{s+N-1})$, being "close" together is higher than the Nth power of the probability of any two points, x_t and x_s, being together. This means that some "clustering" is occurring too frequently in an N-dimensional space. In other words, some "patterns" of exchange rate movements occur more frequently than would be predicted had the data been truly random.

One final point to note is that the BDS statistic tests the null hypothesis of a *random* independent and identically distributed system. A rejection of this null hypothesis is consistent with some type of dependence in the data, which could result from a linear stochastic system, a nonlinear stochastic system, or a nonlinear deterministic system. Additional diagnostic tests are needed to determine the source of rejection.

It is appropriate here to discuss other tests of nonlinearity. McLeod and Li (1983) show that the autocorrelation coefficients and Box-Pierce Q-statistics of the squared residuals of an ARMA model can be used to test for nonlinear dependence. This method, of course, can be applied to the raw data. To avoid confusion, I use $\rho_x(k)$ and $\rho_{xx}(k)$ to denote the kth autocorrelation coefficient of $\{x_t\}$ and $\{x_t^2\}$, and $Q_x(K)$ and $Q_{xx}(K)$ to denote the Box-Pierce Q-statistic for the first K autocorrelations of $\{x_t\}$ and $\{x_t^2\}$. The $Q_{xx}(K)$ statistic is clearly related to Engle's (1982) test for heteroskedasticity since the former uses the autocorrelation coefficients of $\{x_t^2\}$ while the latter uses their partial autocorrelation coefficients.

Tsay (1986) proposes a different test of nonlinearity. This is a more powerful generalization of Keenan's (1985) test. The Tsay test is computed in three steps: (*a*) Regress x_t on the vector $w_t = (1\, x_{t-1} \ldots x_{t-M})'$ and save the residuals u_t. (*b*) Regress the vector $z_t = (x_{t-1}^2\, x_{t-1}x_{t-2} \ldots x_{t-M}^2)'$ on w_t and save the residual vector V_t. Note that z_t is an $M(M + 1)/2$ vector of the unique elements of the cross products $\{x_{t-i}x_{t-j},\ \ i \neq j, i, j = 1, \ldots, M\}$. (*c*) Regress u_t on V_t, saving the residual v_t, and form the statistic

$$\zeta = \{[\Sigma\, V_t u_t]\, [\Sigma\, V_t' V_t]^{-1}\, [\Sigma\, V_t' u_t]/m\}/[\Sigma\, v_t^2/(T - M - m - 1)], \quad (3.7)$$

where $m = M(M + 1)/2$. The limiting distribution of ζ is $F(m, T - M - m - 1)$. Simulations in Tsay (1986) show that this test has good power against the nonlinear moving average and the bilinear models. It is, however, possible that the Tsay test has low power against ARCH models. Consider the simplest ARCH process

$$x_t = \mu + \epsilon_t, \quad (3.8)$$

where ϵ_t (conditional on past information) is normal, with zero mean and variance $h_t = \alpha + \phi x_{t-1}^2$, for $\alpha > 0, 0 < \phi < 1$. It suffices to choose $M = 2$. In step *a*, I regress x_t on $(1\ x_{t-1}\ x_{t-2})$. Asymptotically, the residuals will be ϵ_t. In step *b*, I regress x_{t-1}^2, $x_{t-1}\ x_{t-2}$, and x_{t-2}^2 on $(1\ x_{t-1}\ x_{t-2})$. Asymptotically, the residuals will be ϵ_{t-1}^2, $\epsilon_{t-1}\ \epsilon_{t-2}$, and ϵ_{t-2}^2. In step *c*, I regress the residuals of step *a* on the vector of residuals of step *b*, that is, asymptotically, I regress ϵ_t on ϵ_{t-1}^2, $\epsilon_{t-1}\ \epsilon_{t-2}$, and ϵ_{t-2}^2. We are unlikely to find any significant coefficients, since the mean of ϵ_t conditional on past data is zero.

Another popular nonlinearity test uses the bispectrum, as in Hinich (1982) and Hinich and Patterson (1985). The weakness of this procedure is that it has low power against the class of processes with zero third-order cumulants. It turns out that Engle's (1982) ARCH process is one such candidate. I therefore omit the bispectrum test because ARCH is a very good description of the data.

IV. Application to Foreign Exchange Rates

The data consist of daily closing bid prices of foreign currencies in terms of U.S. dollars from the interbank market provided by the University of Chicago Center for Research on Security Prices. Five major currencies are used: British pound (BP), Canadian dollar (CD), Deutsche mark (DM), Japanese yen (JY), and Swiss franc (SF). There are a total of 2,510 daily observations, from January 2, 1974, to December 30, 1983. The rates of change are calculated by taking the logarithmic differences between successive trading days.

Table 1 provides summary statistics of the data. All five currencies have very heavy tails. The kurtosis coefficients are all substantially larger than that of the standard normal distribution (which is 3). Table 2 gives the autocorrelation of the five currencies. As expected, there is little linear dependence in the data. Standard errors and the Box-Pierce

TABLE 1 **Summary Statistics of Log Price Changes, 1974–83**
$\log(S_t/S_{t-1}) \cdot 100$

	BP	CD	DM	JY	SF
Mean	−.0184	−.0089	.0005	.0077	.0171
Median	.0000	.0098	.0000	.0000	.0000
SD	.5921	.2234	.6372	.6260	.7889
Skewness	−.4136	−.3149	−.4249	−.2044	−.2835
Kurtosis	8.90	8.61	12.79	11.27	10.22
Maximum	3.7496	1.5492	3.6686	3.5703	4.4466
Minimum	−4.6623	−1.8677	−7.0967	−6.2566	−7.0054
Runs test:	1.96	−.86	2.38	.64	.77
N(0,1)	(.0500)	(.3898)	(.0173)	(.5222)	(.4413)

NOTE.—Marginal significance level (two-tailed test) is in parentheses. Number of observations = 2,510. See text for definition of abbreviations.

TABLE 2 **Autocorrelation Coefficient of Log Price Changes, 1974–83 (Heteroskedasticity-consistent SEs)**

Lags	BP	CD	DM	JY	SF
$\rho_x(1)$	−.0216	.0406	−.0638*	−.0569	−.0410
	[.0286]	[.0354]	[.0274]	[.0254]	[.0311]
$\rho_x(2)$	−.0025	.0201	.0021	.0237	.0046
	[.0279]	[.0267]	[.0351]	[.0260]	[.0289]
$\rho_x(3)$	−.0075	.0176	.0240	.0344	.0050
	[.0248]	[.0274]	[.0261]	[.0245]	[.0265]
$\rho_x(4)$	−.0064	.0317	−.0244	.0055	−.0229
	[.0237]	[.0282]	[.0261]	[.0249]	[.0270]
$\rho_x(5)$	.0224	.0740*	.0265	.0386	−.0074
	[.0275]	[.0254]	[.0289]	[.0265]	[.0290]
$\rho_x(6)$	−.0021	.0212	.0373	.0140	.0561
	[.0254]	[.0254]	[.0251]	[.0240]	[.0252]
$\rho_x(7)$	−.0124	.0306	.0113	−.0094	−.0070
	[.0221]	[.0268]	[.0235]	[.0218]	[.0241]
$\rho_x(8)$	−.0155	−.0065	.0128	.0093	−.0290
	[.0226]	[.0240]	[.0248]	[.0235]	[.0233]
$\rho_x(9)$	.0586*	.0483	.0391	.0739*	.0385
	[.0218]	[.0237]	[.0232]	[.0238]	[.0257]
$\rho_x(10)$	.0017	.0266	.0333	.0715*	.0296
	[.0253]	[.0221]	[.0225]	[.0245]	[.0273]
$\rho_x(20)$	.0247	−.0238	−.0061	.0067	−.0099
	[.0214]	[.0216]	[.0239]	[.0252]	[.0241]
$\rho_x(30)$	−.0033	−.0405	−.0203	.0095	−.0004
	[.0237]	[.0217]	[.0249]	[.0234]	[.0222]
$\rho_x(40)$	.0197	−.0083	.0271	.0566*	−.0053
	[.0236]	[.0239]	[.0221]	[.0230]	[.0233]
$\rho_x(50)$	.0170	.0170	.0206	−.0267	.0029
	[.0224]	[.0212]	[.0222]	[.0216]	[.0258]
Adjusted Box-Pierce $Q_x(50)$	39.71	54.35	44.85	70.27	44.72
	(.8512)	(.3123)	(.6796)	(.0308)	(.6846)

NOTE.—Marginal significance levels are in parentheses. Heteroskedasticity-consistent SEs are in brackets. Abbreviations are defined in text.

* Significantly different from zero at the 1% level (one-tailed test).

Q-statistics are adjusted for heteroskedasticity according to Diebold (1988). They show little serial correlation. Nonparametric tests, such as the runs test, also fail to detect any linear dependence.

Table 3 reports the BDS statistics, which indicate substantial nonlinear dependence in the data. In computing the BDS statistics, I have two important issues to deal with: the choice of ℓ and N, and the small sample properties of the BDS statistics. These issues are intimately related. For a given N, ℓ cannot be too small because $C_N(\ell, T)$ will capture too few points; also ℓ cannot be too large because $C_N(\ell, T)$ will capture too many points. For my purposes, ℓ is set in terms of the standard deviation of the data, that is, $\ell = 1$ means that it is one standard deviation of the data.

TABLE 3 **BDS Test: Raw Data**

N	ℓ	BP	CD	DM	JY	SF
2	1.50	8.71	11.12	9.41	8.64	9.34
3	1.50	11.68	13.91	12.52	11.47	12.62
4	1.50	13.41	16.27	15.16	13.73	15.31
5	1.50	14.99	18.26	17.37	15.89	17.46
6	1.50	16.61	20.18	19.62	18.42	19.55
7	1.50	18.21	22.09	21.43	20.70	21.43
8	1.50	19.65	24.10	23.40	23.11	23.33
9	1.50	21.12	26.06	25.46	25.57	25.45
10	1.50	22.89	28.08	27.65	28.25	27.93
2	1.25	9.76	11.89	9.53	9.92	9.88
3	1.25	13.17	14.87	12.84	13.63	13.73
4	1.25	15.38	17.47	15.88	16.52	17.00
5	1.25	17.60	19.80	18.71	19.50	20.00
6	1.25	19.93	22.42	21.70	23.05	22.87
7	1.25	22.43	25.26	24.38	26.66	25.77
8	1.25	25.08	28.40	37.45	30.89	29.11
9	1.25	27.95	31.74	30.81	35.69	33.01
10	1.25	31.50	35.42	34.66	41.37	37.88
2	1.00	10.73	12.61	9.86	10.95	10.82
3	1.00	14.74	15.75	13.70	15.60	15.24
4	1.00	17.93	18.75	17.32	19.72	19.38
5	1.00	21.57	21.78	21.11	24.61	23.44
6	1.00	25.65	25.59	25.50	30.74	28.07
7	1.00	30.44	30.06	29.97	38.05	33.25
8	1.00	36.42	35.28	35.48	47.62	39.92
9	1.00	43.75	41.36	42.11	59.81	48.50
10	1.00	53.30	48.65	50.36	75.96	59.93
2	.75	11.63	12.95	10.34	13.09	11.69
3	.75	16.76	16.07	14.83	19.73	16.89
4	.75	21.69	19.53	19.46	26.66	22.44
5	.75	28.16	23.57	25.07	36.25	28.62
6	.75	36.57	28.95	32.55	49.38	36.64
7	.75	48.26	35.57	41.62	67.95	46.96
8	.75	65.76	43.87	54.11	95.26	62.00
9	.75	91.92	54.55	71.52	135.91	83.66
10	.75	131.14	68.65	95.66	195.80	115.51
2	.50	14.69	12.65	10.84	17.23	12.81
3	.50	22.67	15.70	16.62	27.81	19.45
4	.50	33.16	19.91	23.24	42.66	27.88
5	.50	50.21	25.02	32.93	67.55	37.24
6	.50	78.63	32.20	48.67	110.04	52.89
7	.50	131.65	41.97	72.22	184.82	76.23
8	.50	231.40	55.74	112.48	321.97	113.37
9	.50	433.09	74.92	186.11	580.91	173.69
10	.50	843.42	105.57	317.35	1062.04	279.16

NOTE.—See text for definition of terms.

I present the BDS statistics for $\ell = 1.5, 1.25, 1.0, .75$, and $.50$ and $N = 2, 3, \ldots, 10$ for the raw data. It is clear that the BDS statistics all lie in the extreme positive tail of the standard normal distribution. The data strongly reject the null hypothesis of iid. Note that the BDS statistics for low dimensions (e.g., $N = 2, 3, 4$) are not sensitive to the choice of ℓ, but those for higher dimensions (e.g., $N = 8, 9, 10$) are. The reason is that, while I have 1,250 nonoverlapping 2-histories at dimension 2, I only have 250 nonoverlapping 10-histories at dimension 10.

Since the BDS statistics in table 3 are so large, one might question whether they are reasonable. The large BDS statistics can arise in two ways: either the finite sample distribution under the null hypothesis of iid is poorly approximated by the asymptotic normal distribution, or the BDS statistics are large when the null hypothesis of iid is violated.

Monte Carlo evidence indicates that the BDS statistic for my data can reliably be approximated by its asymptotic distribution. Table 4 presents bootstrap experiments using the data in this study. I compute the BDS statistics for 625 random permutations of the ordering of the data. (There are 2,510! orderings, which is an enormous number. The bootstrap results are essentially the same if we sample with replacement.) The mean BDS statistic is close to zero, and the standard deviation is close to one. At low dimensions, it is normally distributed. The observed ordering of the data, in contrast, yields BDS statistics that are larger than those from all 625 random permutations. This clearly shows that the observed ordering is not randomly sampled from the 2,510! permutations.

Tables 5–7 present Monte Carlo evidence regarding the size of the BDS statistic under the null hypothesis of iid, performed in Hsieh and LeBaron (1988*a*). These tables indicate that, at sample sizes of 1,000, asymptotic normality is appropriate for data drawn from the standard normal, Student-t with 3 degrees of freedom, double exponential, and chi-square with 4 degrees of freedom, when ℓ is between 1 and 2, and that asymptotic normality is appropriate for data drawn from the uniform and bimodal distributions only when ℓ is between 1.5 and 2. However, the first four distributions are most relevant for my data, which are unimodal and heavy tailed.

Table 8 gives the distribution of the BDS statistic under five alternatives: autoregression of order 1 (AR1), moving average of order 1 (MA1), nonlinear moving average, ARCH of order 1, threshold autoregressive model, and the tent map, performed in Hsieh and LeBaron (1988*b*). They show that, for samples of 1,000 observations, the average BDS statistic is around 10 and can become as large as 185. If the sample size is enlarged from 1,000 to 2,500, these averages should increase by another 50%. Hence the BDS statistics reported in table 3 are not "usually" large when compared to these alternative models.

TABLE 4 **Distribution of BDS Statistics at ℓ = 1 (625 Random Samples of 2,510 Points without Replacement)**

N	BP	CD	DM	JY	SF
2:					
Mean	−.04	−.04	−.01	.07	−.04
SD	1.05	1.01	1.04	.96	1.07
Skewness	.15	−.14	−.07	.00	.09
Kurtosis	2.92	2.98	2.91	2.72	2.79
3:					
Mean	−.07	−.05	−.05	.07	−.05
SD	1.05	.99	1.01	.94	1.03
Skewness	.06	−.10	−.11	.15	.07
Kurtosis	2.87	2.86	2.97	2.94	2.75
4:					
Mean	−.06	−.03	−.05	.06	−.07
SD	1.05	.99	1.01	.94	1.04
Skewness	.11	.04	−.07	.22	.07
Kurtosis	3.00	2.97	2.82	3.00	2.79
5:					
Mean	−.04	−.02	−.04	.05	−.06
SD	1.05	.98	1.02	.95	1.04
Skewness	.24	.12	−.08	.24	.06
Kurtosis	3.43	2.96	2.85	3.06	2.76
6:					
Mean	−.04	−.01	−.03	.05	−.05
SD	1.05	.98	1.02	.95	1.05
Skewness	.36	.20	−.05	.23	.10
Kurtosis	3.81	2.89	2.78	3.14	2.70
7:					
Mean	−.03	−.01	−.03	.05	−.06
SD	1.04	.97	1.02	.96	1.05
Skewness	.49	.25	.03	.24	.15
Kurtosis	4.21	2.93	2.67	3.15	2.70
8:					
Mean	−.02	−.01	−.02	.05	−.05
SD	1.04	.97	1.02	.97	1.06
Skewness	.62	.31	.11	.31	.24
Kurtosis	4.74	2.93	2.62	3.28	2.75
9:					
Mean	.00	−.01	.00	.05	−.04
SD	1.05	.97	1.01	.97	1.07
Skewness	.74	.39	.17	.42	.32
Kurtosis	5.30	3.01	2.63	3.47	2.81
10:					
Mean	.02	−.02	.00	.06	−.04
SD	1.06	.99	1.01	.97	1.08
Skewness	.85	.46	.24	.47	.40
Kurtosis	5.96	3.12	2.66	3.51	2.88

NOTE.—Abbreviations are defined in text.

TABLE 5 Size of BDS Statistics at Dimension 2 (2,000 Replications; 1,000 Points per Replication)

	ℓ					Nominal Size
	.25	.50	1.00	1.50	2.00	
Standard normal:						
% < −2.33	4.65	1.40	1.05	.90	.80	1.00
% < −1.96	8.95	3.25	2.90	2.45	2.65	2.50
% > 1.96	6.30	3.70	2.25	2.40	2.50	2.50
% > 2.33	3.60	1.55	.90	.70	.90	1.00
$t(3)$:						
% < −2.33	1.25	.65	.85	.50	.40	1.00
% < −1.96	3.25	2.50	2.20	2.05	1.20	2.50
% > 1.96	4.15	3.10	2.80	3.20	3.55	2.50
% > 2.33	1.90	1.50	1.10	1.45	1.80	1.00
Double exponential:						
% < −2.33	1.25	.95	.75	.70	.60	1.00
% < −1.96	3.10	2.75	2.85	2.40	2.30	2.50
% > 1.96	3.30	3.00	3.20	3.30	3.10	2.50
% > 2.33	1.45	1.25	1.45	1.70	1.75	1.00
$\chi^2(4)$:						
% < −2.33	1.65	.90	1.10	1.20	1.10	1.00
% < −1.96	5.00	3.05	3.00	3.35	2.45	2.50
% > 1.96	5.05	3.80	3.85	3.90	3.70	2.50
% > 2.33	3.25	2.10	1.90	1.65	2.15	1.00
Uniform:						
% < −2.33	44.95	21.75	1.45	1.40	1.40	1.00
% < −1.96	46.05	26.60	3.60	3.00	3.15	2.50
% > 1.96	42.45	24.30	5.05	2.85	2.85	2.50
% > 2.33	41.40	21.80	3.10	1.30	1.25	1.00
Bimodal:						
% < −2.33	2.30	2.55	52.70	1.40	1.10	1.00
% < −1.96	5.45	5.00	54.70	3.85	3.10	2.50
% > 1.96	6.45	5.75	29.20	3.85	3.05	2.50
% > 2.33	3.45	3.05	28.10	2.05	1.40	1.00

NOTE.—Approximate SE is 1.12 for these probabilities.

Furthermore, table 8 shows that the BDS has good power against these alternative hypotheses.

The Monte Carlo evidence supports the conclusion that the BDS test strongly rejects the null hypothesis of iid. To ensure that the BDS test is not merely picking up some linear dependence in the data (since table 8 shows that BDS has power against an AR process), I prefilter the data by the following autoregression:

$$x_t = \beta_0 + \beta_M D_{M,t} + \beta_T D_{T,t} + \beta_W D_{W,t} + \beta_R D_{R,t} + \beta_H \mathrm{HOL}_t + \sum_{i=1}^{m} \beta_i x_{t-i} + u_t,$$

where $D_{M,t}$, $D_{T,t}$, $D_{W,t}$, and $D_{R,t}$ are dummy variables for Monday, Tuesday, Wednesday, and Thursday, respectively, and HOL_t is the

TABLE 6 **Size of BDS Statistics at Dimension 5 (2,000 Replications; 1,000 Points per Replication)**

	ℓ					Nominal
	.25	.50	1.00	1.50	2.00	Size
Standard normal:						
% < −2.33	29.85	3.60	.55	.80	.75	1.00
% < −1.96	32.75	7.40	2.35	2.35	2.55	2.50
% > 1.96	29.65	8.15	2.90	2.40	2.50	2.50
% > 2.33	26.95	5.30	1.40	1.10	1.30	1.00
$t(3)$:						
% < −2.33	6.05	.70	.70	.85	.60	1.00
% < −1.96	9.55	2.25	2.30	2.55	2.50	2.50
% > 1.96	11.00	4.20	3.10	3.50	3.55	2.50
% > 2.33	7.55	2.25	1.95	1.70	1.60	1.00
Double exponential:						
% < −2.33	5.90	.35	.75	.95	.75	1.00
% < −1.96	9.65	2.10	2.40	2.60	2.75	2.50
% > 1.96	10.30	3.70	2.90	2.25	2.50	2.50
% > 2.33	7.00	1.75	1.25	.95	1.30	1.00
$\chi^2(4)$:						
% < −2.33	16.15	.95	.85	.85	1.00	1.00
% < −1.96	20.65	3.45	2.30	2.25	2.30	2.50
% > 1.96	19.20	5.30	3.45	3.30	3.00	2.50
% > 2.33	15.40	2.95	1.75	1.25	1.50	1.00
Uniform:						
% < −2.33	49.20	35.50	4.05	1.50	1.30	1.00
% < −1.96	49.60	37.40	7.55	3.00	2.95	2.50
% > 1.96	48.05	38.50	6.85	3.75	3.40	2.50
% > 2.33	47.90	36.85	4.30	1.55	1.25	1.00
Bimodal:						
% < −2.33	15.45	7.25	46.00	2.50	1.45	1.00
% < −1.96	20.00	10.80	47.20	5.70	3.30	2.50
% > 1.96	17.85	10.25	41.05	5.30	2.70	2.50
% >2.33	13.65	6.85	39.95	2.80	1.40	1.00

NOTE.—Approximate SE is 1.12 for these probabilities.

number of holidays (excluding weekends) between two successive trading days. The lag length of the AR(m) model is chosen for which the adjusted $Q_x(50)$ test is not significant at the 10% level. The identified models are $m = 0, 5, 6, 10$, and 6, respectively, for the BP, CD, DM, JY, and SF. An alternative procedure uses 10 lags for all five currencies.

It is appropriate to deal with the induced serial correlation in fitted residuals. Brock (1987) shows that the asymptotic distribution of the BDS statistic applies to residuals of linear regressions as well as the original data.[3] Baek and Brock (1988) extend this result for vector autoregressions.

3. I have verified this in Monte Carlo experiments. The BDS statistics applied to residuals from an AR1 ($\rho = .5$ and .95) and an MA1 ($\theta = .5$ and .95) conform very well to the asymptotic normal distribution, using 2,000 replications of 1,000 observations each.

TABLE 7 **Size of BDS Statistics at Dimension 10 (2,000 Replications; 1,000 Points per Replication)**

	ℓ					Nominal Size
	.25	.50	1.00	1.50	2.00	
Standard normal:						
% < −2.33	99.85	39.35	.80	.60	1.05	1.00
% < −1.96	99.85	39.60	2.55	1.80	2.65	2.50
% > 1.96	.15	34.05	5.00	3.15	2.70	2.50
% > 2.33	.15	32.90	3.65	1.55	1.30	1.00
t(3):						
% < −2.33	95.00	5.45	.40	.85	.90	1.00
% < −1.96	95.15	9.60	1.35	2.55	2.50	2.50
% > 1.96	3.70	12.60	3.90	3.05	3.15	2.50
% > 2.33	3.65	9.40	2.00	1.70	1.45	1.00
Double exponential:						
% < −2.33	49.45	6.90	.25	.65	1.10	1.00
% < −1.96	95.90	12.50	1.00	2.00	2.80	2.50
% > 1.96	1.00	13.80	3.70	2.80	2.25	2.50
% > 2.33	1.00	11.15	2.15	1.25	1.30	1.00
$\chi^2(4)$:						
% < −2.33	98.95	19.10	.30	.50	.55	1.00
% < −1.96	98.95	24.35	1.55	1.85	2.15	2.50
% > 1.96	1.05	20.90	4.60	3.65	2.85	2.50
% > 2.33	1.05	17.35	2.65	1.75	1.40	1.00
Uniform:						
% < −2.33	99.85	51.60	16.30	.90	.95	1.00
% < −1.96	99.85	51.95	20.85	3.40	2.50	2.50
% > 1.96	.15	42.30	18.70	4.40	3.55	2.50
% > 2.33	.15	41.20	15.50	2.50	1.70	1.00
Bimodal:						
% < −2.33	93.10	35.50	47.95	5.80	.90	1.00
% < −1.96	93.10	39.15	48.40	9.75	2.90	2.50
% > 1.96	6.90	32.55	47.00	8.85	3.10	2.50
% > 2.33	6.90	30.45	46.55	5.30	1.40	1.00

NOTE.—Approximate SE is 1.12 for these probabilities.

Table 9 reports the BDS test of the two sets of filtered data for $\ell = 1$. They do not differ substantially from those using the raw data, which suggests that the BDS test is not merely picking up some linear dependence but is in fact detecting strong *nonlinear* dependence in the data.

Table 10 gives the results of other tests of nonlinearity of the raw data. The autocorrelation coefficients of the squared data, $\rho_{xx}(k)$, and the Ljung-Box $Q_{xx}(K)$ of the squared data are both substantially larger than the corresponding $\rho_x(k)$ and $Q_x(K)$ in table 2, corroborating the BDS inference that the data contain important nonlinearity. Interestingly, the Tsay test picks up little nonlinearity when $M = 2$ or 4, except for the CD. But at $M = 8$ and 10, the Tsay test is able to detect some nonlinearity.

TABLE 8 **Distribution of BDS Statistics for Various Alternatives (2,000 Replications of 1,000 Points)**

N	AR1 $\rho = 0.5$	MA1 $\theta = 0.5$	Nonlinear MA	ARCH $\phi = 0.5$	Threshold AR	Tent Map
Mean (SD) at $\ell = 1.00$:						
2	15.98	9.68	6.90	12.26	5.84	114.90
	(2.46)	(1.76)	(1.55)	(1.70)	(1.51)	(3.25)
3	14.92	9.47	9.08	11.98	5.47	111.93
	(2.46)	(1.99)	(1.57)	(1.85)	(1.48)	(4.24)
4	13.95	8.94	9.57	11.39	4.99	115.94
	(2.45)	(1.66)	(1.65)	(1.96)	(1.46)	(4.75)
5	13.25	8.46	9.68	10.88	4.60	121.02
	(2.51)	(1.69)	(1.77)	(2.09)	(1.46)	(5.49)
6	12.76	8.07	9.66	10.48	4.29	128.69
	(2.60)	(1.73)	(1.91)	(2.23)	(1.47)	(6.46)
7	12.42	7.77	9.62	10.19	4.04	138.68
	(2.73)	(1.81)	(2.07)	(2.40)	(1.50)	(7.78)
8	12.21	7.53	9.59	9.98	3.83	151.24
	(2.90)	(1.91)	(2.26)	(2.60)	(1.55)	(9.53)
9	12.11	7.35	9.60	9.85	3.66	166.61
	(3.12)	(2.04)	(2.47)	(2.84)	(1.62)	(11.78)
10	12.08	7.20	9.63	9.78	3.51	185.24
	(3.38)	(2.19)	(2.72)	(3.10)	(1.71)	(14.73)
% of replications rejected at 1% (two-tailed) level at $\ell/\sigma = 1$:						
2	100.00	100.00	100.00	100.00	98.90	100.00
3	100.00	100.00	100.00	100.00	98.45	100.00
4	100.00	100.00	100.00	100.00	95.65	100.00
5	100.00	99.95	100.00	100.00	92.00	100.00
6	100.00	99.95	100.00	100.00	87.65	100.00
7	100.00	99.95	100.00	100.00	83.65	100.00
8	100.00	99.90	100.00	100.00	79.05	100.00
9	100.00	99.75	100.00	99.95	74.60	100.00
10	100.00	99.45	100.00	99.90	70.05	100.00

NOTES.—The figures in the table are the percentage of test statistics with absolute value greater than 2.576. One SE bound is 1.12%. Standard errors are in parentheses.

AR1:	$x_t = \rho\, x_{t-1} + \epsilon_t$
MA1:	$x_t = \theta\, \epsilon_{t-1} + \epsilon_t$
Nonlinear MA:	$x_t = \epsilon_t + .8\epsilon_{t-1}\, \epsilon_{t-2}$
ARCH:	$x_t = [1 + \phi x_{t-1}^2]^{.5}\epsilon_t$
Threshold AR:	$x_t = -.5\, x_{t-1} + \epsilon_t$, if $x_{t-1} < .5$
	$= .4\, x_{t-1} + \epsilon_t$, if $x_{t-1} \geq .5$
Tent:	$x_t = 2\, x_{t-1}$, if $x_{t-1} < .5$
	$= 2 - 2\, x_{t-1}$, if $x_{t-1} \geq .5$

TABLE 9 BDS Test: Filtered Data

N	ℓ		BP	CD	DM	JY	SF
		lags =	0	5	6	10	6
2	1.00		11.09	12.37	8.60	10.60	10.28
3	1.00		15.00	15.43	12.82	15.29	14.73
4	1.00		18.06	18.06	16.42	19.72	18.78
5	1.00		21.56	20.85	20.32	24.77	22.82
6	1.00		25.56	24.32	24.69	30.78	27.35
7	1.00		30.01	28.21	29.25	37.94	32.46
8	1.00		35.63	32.59	34.78	47.33	38.98
9	1.00		42.65	37.59	41.37	59.31	47.21
10	1.00		51.57	43.75	49.57	75.21	58.20
		lags =	10	10	10	10	10
2	1.00		10.51	12.35	8.63	10.60	10.26
3	1.00		14.71	15.40	12.83	15.29	14.75
4	1.00		18.01	18.03	16.55	19.72	18.87
5	1.00		21.63	20.68	20.49	24.77	22.92
6	1.00		25.63	24.02	24.83	30.78	27.46
7	1.00		30.35	27.73	29.34	37.94	32.63
8	1.00		36.19	31.93	34.81	47.33	39.27
9	1.00		43.20	36.81	41.37	59.31	47.63
10	1.00		52.20	42.85	49.53	75.21	58.79

NOTE.—All test statistics are significant at the 1% level. Abbreviations are defined in text.

V. Discriminating between Different Types of Nolinearities

In this section, I try to sort out the type of nonlinearity in the data. Let u_t denote the linearly filtered data, that is, u_t is the residual from the autoregression with 10 lags and daily dummies in the previous section. We can distinguish between two types of nonlinear dependence in u_t:

Additive dependence:

$$u_t = v_t + f(x_{t-1}, \ldots, x_{t-k}, u_{t-1}, \ldots, u_{t-k}); \tag{5.1}$$

Multiplicative dependence:

$$u_t = v_t f(x_{t-1}, \ldots, x_{t-k}, u_{t-1}, \ldots, u_{t-k}), \tag{5.2}$$

where v_t is an iid random variable with zero mean and independent of past x_t's and u_t's, and $f(\ )$ an arbitrary nonlinear function of $x_{t-1}, \ldots, x_{t-k}$, and $u_{t-1}, \ldots, u_{t-k}$, for some finite k. Additive dependence postulates that nonlinearity enters only through the mean of the process, which is closely related to Priestley's state-dependent model (1980; see eq. [3.14], p. 53). The nonlinear moving average, the threshold autoregression, and the bilinear model are examples of additive dependence. Multiplicative dependence postulates that nonlinearity enters only through the variance of the process, which is essentially the general form of conditional heteroskedasticity in Engle (1982; see eq.

TABLE 10 Some Standard Tests for Nonlinearity: Raw Data

Lag	BP	CD	DM	JY	SF
Autocorrelation coefficients of squared log price changes:					
$\rho_{xx}(1)$	.1333	.2806	.0753	.0603	.1549
$\rho_{xx}(2)$	.1200	.1024	.1773	.0674	.1192
$\rho_{xx}(3)$	.0677	.1160	.0609	.0494	.0835
$\rho_{xx}(4)$	.0522	.1290	.0607	.0538	.0899
$\rho_{xx}(5)$	.1137	.0804	.0931	.0746	.1218
$\rho_{xx}(6)$	.0776	0805	.0490	.0433	.0647
$\rho_{xx}(7)$	.0291	.1041	.0328	.0192	.0492
$\rho_{xx}(8)$	.0355	.0588	.0464	.0371	.0390
$\rho_{xx}(9)$	.0240	.0534	.0302	.0412	.0721
$\rho_{xx}(10)$	.0756	.0288	.0228	.0495	.0948
Ljung-Box					
$Q_{xx}(50)$	365.84*	593.36*	215.06*	206.09*	531.87*
	(.0000)	(.0000)	(.0000)	(.0000)	(.0000)
Tsay test for nonlinearity:					
$M = 2$:	2.19	4.13*	.62	.74	1.46
$F(3,2504)$	(.0857)	(.0064)	(.6061)	(.5315)	(.2221)
$M = 4$:	2.04	4.95*	1.96	1.74	1.17
$F(10,2495)$	(.0261)	(.0000)	(.0338)	(.0666)	(.3062)
$M = 6$:	1.55	4.64*	2.30*	2.05*	1.71
$F(28,2474)$	(.0524)	(.0000)	(.0007)	(.0034)	(.0231)
$M = 8$:	1.67*	3.25*	2.05*	2.05*	1.42
$F(36,2465)$	(.0076)	(.0000)	(.0002)	(.0002)	(.0194)
$M = 10$:	2.15*	2.51*	2.09*	2.24*	1.64*
$F(55,2444)$	(.0000)	(.0000)	(.0000)	(.0000)	(.0023)

NOTE.—Marginal significance levels are in parentheses. Abbreviations are defined in text.
* Significantly different from zero at the 1% level (one-tailed test).

[5], p. 989). Note that the ARCH-M (i.e., ARCH-in-the-mean) model, used in Domowitz and Hakkio (1985) and Diebold and Pauly (1988), is a hybrid since nonlinearity enters both the mean and the variance.

Both additive and multiplicative nonlinearity imply that u_t^2 is correlated with its own lags. This, of course, is evident in the autocorrelation coefficients of the squared raw data in table 10. However, multiplicative dependence implies that

$$E[u_t | x_{t-1}, \ldots, x_{t-k}, u_{t-1}, \ldots, u_{t-k}] = 0, \qquad (5.3)$$

while additive dependence implies that

$$E[u_t | x_{t-1}, \ldots, x_{t-k}, u_{t-1}, \ldots, u_{t-k}] \neq 0. \qquad (5.4)$$

I can exploit this distinction to discriminate between the two types of nonlinearity. Suppose $f(\ )$ is at least twice continuously differentiable. I can approximate it by a second-order Taylor series expansion around zero and obtain terms such as $u_{t-i}u_{t-j}$, $x_{t-i}u_{t-j}$, and $x_{t-i}x_{t-j}$. Multiplicative dependence implies that u_t is not correlated with these terms, while additive dependence implies that u_t is usually correlated with at least some of these terms. (Note that Pemberton and Tong [1981] give examples of nonlinear autoregressive models whose odd product moments are zero.)

To implement a test, I define $\rho_{uuu}(i, j) = E(u_t u_{t-i} u_{t-j})/\rho_u^3$. Set up multiplicative nonlinearity as the null hypothesis, which implies $\rho_{uuu}(i, j) = 0$ for all $i, j > 0$, and test it against the composite alternative hypothesis that $\rho_{uuu}(i, j) \neq 0$ for some $i, j > 0$. I estimate $\rho_{uuu}(i, j)$ by

$$r_{uuu}(i, j) = \left[\frac{1}{T} \Sigma\, u_t u_{t-i} u_{t-j}\right] \Big/ \left[\frac{1}{T} \Sigma\, u_t^2\right]^{1.5}. \qquad (5.5)$$

Under the null hypothesis of $\rho_{uuu}(i, j) = 0$ and auxiliary assumptions about the behavior of $\{u_t\}$, $\sqrt{T}[(1/T)\, \Sigma\, u_t u_{t-i} u_{t-j}]$ is asymptotically normally distributed, with mean zero and variance $\omega(i, j) = \text{plim}_{T\to\infty}(1/T)\, \Sigma\, u_t^2 u_{t-i}^2 u_{t-j}^2$, provided that the probability limit exists. Then $r_{uuu}(i, j)$ is asymptotically normally distributed, with mean zero and variance $\omega(i, j)/\sigma_u^6$, which can be consistently estimated by $[(1/T)\, \Sigma\, u_t^2 u_{t-i}^2 u_{t-j}^2]/[(1/T)\, \Sigma\, u_t^2]^3$. An asymptotic test of $\rho_{uuu}(i, j) = 0$ can then be obtained.

This procedure is very similar to the Tsay (1986) test for nonlinearity. I test $\rho_{uuu}(i, j) = 0$ individually, while Tsay (1986) tests jointly for $\rho_{uuu}(i, j) = 0$ for $0 < i, j < k$. There is, however, an important difference. Tsay (1986) assumes that u_t is iid, while I assume that $E[u_t | u_{t-1}, \ldots, u_{t-k}] = 0$ along with sufficient moment conditions to guarantee the asymptotic normality of $r_{uuu}(i, j)$ and consistent estimation of its variance. The reason is that the Tsay (1986) test is designed to detect any type of nonlinearity, whether it is additive or multiplicative. My test is designed to reject only in the presence of additive nonlinearity but not multiplicative nonlinearity. However, it should be clear that the Tsay test will have good power only against additive nonlinearity (when $\rho_{uuu}(i, j) \neq 0$ for some i, j), but it will have low power against multiplicative nonlinearity (when $\rho_{uuu}(i, j) = 0$ for all i, j).

To check that the third-order moment test has power against additive nonlinearity, I apply this test to the alternative models in table 8. The results are reported in table 11. The statistic $\sqrt{T}\, r_{uuu}(i, j)/[\omega(i, j)/\sigma_u^6]^{0.5}$ is used to test whether third-order moments are different from zero, up to the fourth lag. A rejection is registered if the absolute value of this statistic is larger than 2.576, constituting a two-tailed test at the 1% significance level. The test rejects the null hypothesis of zero third-order moments at approximately the nominal size of 1% for the AR1, the MA1, and the ARCH models, which have no additive nonlinearity. The test also rejects the null hypothesis of zero third-order moments at about 99% for the nonlinear moving average (at $i = 2$, $j = 1$), the threshold autoregression (at $i = 1, j = 1$), and the tent map (at $i = 1, j$

TABLE 11 Power of Third-Order Moment Tests for Different Models (2,000 Replications of 1,000 Points), % of Replications Rejected at 1% Level (Two-Tailed Test)

Lag i	Lag j	AR1 $\rho = 0.5$	MA1 $\theta = 0.5$	Nonlinear MA	ARCH $\phi = 0.5$	Threshold AR	Tent Map	ARCH-M $\phi = .5, \delta = 1$
1	1	.05	.00	.05	.10	100.00	100.00	64.00
2	1	.80	.50	99.85	.25	1.30	1.05	.45
2	2	.15	.25	.25	.10	.55	99.80	10.50
3	1	1.65	.85	1.05	.65	1.35	1.45	.50
3	2	1.55	.85	1.05	.40	1.10	1.00	.45
3	3	.35	.25	.05	.40	.15	3.20	.95
4	1	1.15	.75	1.00	.75	1.20	.70	.50
4	2	1.80	1.35	1.70	.85	1.15	1.30	.75
4	3	1.80	.50	1.20	1.00	.80	1.05	.55
4	4	.30	.25	.25	.25	.15	.05	.20

NOTES.—See table 8.
ARCH-M: $x_t = \delta h_t^{1/2} + \epsilon_t$, $V[\epsilon_t] = h_t = 1 + \phi\epsilon_{t-1}^2$

= 1, and $i = 2, j = 2$), which have additive nonlinearity. This shows that the third-order moment test has good power against additive nonlinearity. In addition, the third-order moment test is able to detect additive nonlinearity in a hybrid model, rejecting 64% for the ARCH-M (at $i = 1$ and $j = 1$). Although this rejection rate is low, the power rises to 84% when the sample size increases to 2,500 observations.

Table 12 reports $r_{uuu}(i, j)$ and their standard errors when $\{u_t\}$ is the residual from the tenth-order autoregression with dummies for days of the week and holidays. None of them are significantly different from zero at the 1% level. Although the table reports only the results for $i = j = 5$, none of the third-order moments up to $i = j = 10$, other than $\rho_{uuu}(5, 6)$ for the CD, $\rho_{uuu}(2, 10)$ for the JY and $\rho_{uuu}(6, 10)$ for the SF, is significantly different from zero at the 1% level. Furthermore, the re-

TABLE 12 Third-Order Moments of Filtered Data

Lag *i*	Lag *j*	BP	CD	DM	JY	SF
1	1	−.124	−.132	−.086	−.060	−.103
		(.105)	(.150)	(.081)	(.073)	(.110)
2	1	.008	−.054	−.008	.024	.021
		(.050)	(.065)	(.053)	(.035)	(.048)
2	2	−.109	−.030	.008	.004	.009
		(.104)	(.094)	(.176)	(.074)	(.101)
3	1	−.003	−.112	.048	−.011	−.016
		(.042)	(.046)	(.052)	(.033)	(.052)
3	2	−.012	.006	−.030	−.013	−.015
		(.042)	(.055)	(.052)	(.032)	(.047)
3	3	−.001	.176	−.014	.031	−.069
		(.074)	(.114)	(.079)	(.075)	(.103)
4	1	−.034	−.104	−.005	.032	.005
		(.040)	(.054)	(.037)	(.030)	(.049)
4	2	.040	.017	.067	−.009	−.055
		(.045)	(.041)	(.061)	(.041)	(.057)
4	3	.068	.078	.032	.045	−.044
		(.031)	(.079)	(.031)	(.032)	(.038)
4	4	−.026	.171	−.095	.094	−.069
		(.067)	(.109)	(.106)	(.092)	(.091)
5	1	−.019	−.076	.044	.074	.054
		(.047)	(.051)	(.049)	(.036)	(.049)
5	2	−.019	−.020	−.038	−.065	−.035
		(.047)	(.032)	(.067)	(.034)	(.045)
5	3	.009	−.037	.015	−.019	−.031
		(.037)	(.040)	(.054)	(.038)	(.051)
5	4	−.069	−.025	.048	.036	−.021
		(.035)	(.052)	(.039)	(.038)	(.044)
5	5	−.026	.020	−.014	.016	.080
		(.093)	(.068)	(.085)	(.073)	(.119)

NOTE.—Standard errors are in parentheses. Abbreviations are defined in text.

sults do not change if I use the filtered data from the Box-Jenkin identification procedure (i.e., lags of 0, 5, 6, 10, and 6 for the BP, CD, DM, JY, and SF), or if I use the raw data themselves. The evidence supports the view that the changing of variances is responsible for the rejection of iid in exchange rate changes.

In the attempt to model changing variances, I recognize that heteroskedasticity can arise in two ways. An exogenous shift in policy regime can lead to a change in variance of exchange rates. This type of "exogenous" heteroskedasticity will cause the BDS test to reject iid, but it is not what I mean by "nonlinearity" since the change in variance is unpredictable based on past exchange rate changes. However, variance changes can arise endogenously and persist over time, as in ARCH processes. This type of conditional heteroskedasticity will also cause the BDS test to reject iid, and it is an example of a nonlinear time-series model since the change in variance is predictable based on past exchange rate changes.

If heteroskedasticity in exchange rates arises exogenously, it would be (by definition) difficult to model without knowing what the exogenous variables are. A full treatment is outside the scope of this article. I use a simplistic model of "exogenous" heteroskedasticity by assuming that the exogenous variable is time. Suppose that exchange rate changes within a month have the same mean and variance, which can change across months. To test this hypothesis, I apply the following transformation to standardize the data. For each month, the sample mean and standard deviation are computed. Each observation is standardized by subtracting the monthly mean and dividing by the monthly standard deviation. This method removes most of the skewness (except for the JY) and a large part of leptokurtosis in the data. In fact, this transformation should remove the rejection of iid. The results are mixed. The BDS statistics in table 13 are substantially lower than those in table 9. As a matter of fact, the CD and SF now pass the BDS test. But the BP, DM, and JY still fail the test. The $Q_{xx}(50)$ statistic detects nonlinear dependence in the BP, DM, JY, and SF. Furthermore, the runs test and the adjusted $Q_x(50)$ pick up linear dependence. Only the Tsay test fails to find any nonlinearity. This evidence suggests that the simple model of "exogenous" heteroskedasticity is not adequate to describe the data.

For the remainder of this article, I concentrate on modeling conditional heteroskedasticity, using Engle's (1982) ARCH model and Bollerslev's (1986) generalized ARCH (GARCH) model. Applications of ARCH and GARCH to exchange rates can be found in Bollerslev (1987), Diebold (1988), Diebold and Nerlove (1986), Diebold and Pauly (1988), Engle and Bollerslev (1986), Hsieh (1987), Manas-Anton (1986), and Milhøj (1987). I estimate the simplest GARCH model, which is

TABLE 13 Tests of Nonlinearity: Standardized Data

N	ℓ	BP	CD	DM	JY	SF
BDS Tests:						
2	1.00	2.28	.65	3.04*	2.52*	1.55
3	1.00	2.77*	.19	2.85*	3.67*	1.64
4	1.00	3.05*	.24	2.75*	4.74*	1.84
5	1.00	3.52*	.15	2.79*	5.50*	1.45
6	1.00	3.83*	.15	3.08*	6.63*	1.06
7	1.00	4.13*	.05	3.04*	6.75*	.37
8	1.00	4.50*	−.02	3.08*	7.33*	−.09
9	1.00	4.85*	−.16	3.31*	7.98*	−.47
10	1.00	5.30*	−.47	3.46*	8.70*	−.64
Runs test:		1.94	1.38	4.89*	1.58	2.85*
$N(0,1)$		(.0262)	(.0838)	(.0000)	(.0571)	(.0022)
$Q_x(50)$		121.64*	93.80*	117.60*	98.13*	106.53*
		(.0000)	(.0001)	(.0000)	(.0001)	(.0000)
$Q_{xx}(50)$		90.16*	68.82	94.12*	97.31*	78.93*
		(.0000)	(.0399)	(.0000)	(.0000)	(.0056)
Tsay test for nonlinearity:						
$M = 2$:		2.05	.14	1.34	.60	1.76
$F(3,2504)$		(.1032)	(.9324)	(.2585)	(.6191)	(.1509)
$M = 4$:		1.20	1.08	1.17	.98	1.57
$F(10,2495)$		(.2897)	(.3738)	(.3062)	(.4585)	(.1093)
$M = 6$:		1.05	.89	.81	.69	1.26
$F(21,2482)$		(.3973)	(.6054)	(.7107)	(.8474)	(.1905)
$M = 8$:		.96	.95	.83	.76	.88
$F(36,2465)$		(.5373)	(.5545)	(.7529)	(.8480)	(.6735)
$M = 10$:		1.35	.91	.76	.73	.87
$F(55,2444)$		(.0451)	(.6629)	(.9034)	(.9320)	(.7405)
Skewness		−.0789	−.0912	−.0522	.2420	.0240
Kurtosis		3.89	3.19	3.33	3.97	3.42

NOTE.—Abbreviations are defined in text. Marginal significance levels are in parentheses.
* Significant at the 1.0% level (one-tailed test).

specified as follows:

$$x_t = \beta_0 + \beta_M D_{M,t} + \beta_T D_{T,t} + \beta_W D_{W,t} + \beta_R D_{R,t} + \beta_H \mathrm{HOL}_t + \sum_{i=1}^{m} \beta_i x_{t-i} + \epsilon_t, \tag{5.6}$$

where m is 0, 5, 6, 10, and 6, respectively, for the BP, CD, DM, JY, and SF and ϵ_t (conditional on past data) is normally distributed, with zero mean and variance h_t, such that

$$h_t = \gamma_0 + \gamma_M D_{M,t} + \gamma_T D_{T,t} + \gamma_W D_{W,t} + \gamma_R D_{R,t} + \gamma_H \mathrm{HOL}_t + \gamma \mathrm{h}_{t-1} + \phi \epsilon_{t-1}^2. \tag{5.7}$$

TABLE 14 **Tests of Nonlinearity: Standardized Residuals, GARCH(1, 1)-Normal**

N	ℓ	BP	CD	DM	JY	SF
BDS Tests:						
2	1.00	2.94*	2.11	−1.18	−1.03	−.28
3	1.00	3.93*	2.03	−.85	−1.11	.17
4	1.00	4.47*	1.96	−.18	−.65	.46
5	1.00	5.19*	1.86	.67	−.32	.40
6	1.00	6.03*	1.93	1.53	.28	.40
7	1.00	7.17*	1.96	2.03	.70	.15
8	1.00	8.46*	1.95	2.60*	1.25	.02
9	1.00	10.26*	1.75	3.13*	1.85	−.09
10	1.00	12.42*	1.55	3.79*	2.38	−.03
Runs test:		.74	−.86	−1.10	.46	−1.14
$N(0,1)$		(.2297)	(.1949)	(.1357)	(.3228)	(.1271)
$Q_x(50)$		58.25	54.30	56.64	51.56	50.05
		(.0736)	(.0526)	(.0263)	(.0273)	(.0913)
$Q_{xx}(50)$		41.86	40.78	75.94*	38.97	49.13
		(.4771)	(.5245)	(.0010)	(.6047)	(.2091)
Tsay test for nonlinearity:						
M = 2:		.90	.24	.46	2.99	.55
$F(3,2504)$		(.4424)	(.8684)	(.7142)	(.0295)	(.6524)
M = 4:		.79	1.67	.45	1.34	.90
$F(10,2495)$		(.6386)	(.0820)	(.9218)	(.2029)	(.5323)
M = 6:		.76	1.23	.48	1.04	.84
$F(21,2482)$		(.7719)	(.2140)	(.9774)	(.4095)	(.6719)
M = 8:		.85	.83	.64	.83	.72
$F(36,2465)$		(.7220)	(.7529)	(.9528)	(.7678)	(.8915)
M = 10:		.94	.81	.73	.78	.92
$F(55,2444)$		(.6011)	(.8402)	(.9320)	(.8804)	(.6425)
Skewness		−.13	−.06	.002	−.40	.18
Kurtosis		9.91	4.79	6.68	13.23	5.92
Goodness of fit:		357.73*	97.37*	75.71*	355.29*	138.96*
$\chi^2(50)$		(.0000)	(.0001)	(.0109)	(.0000)	(.0000)

NOTE.—Abbreviations are defined in text. Marginal significance levels are in parentheses.
* Significant at the 1.0% level.

After estimation, I perform diagnostic tests on the standarized residuals:

$$z_t = \hat{\epsilon}_t/\hat{h}_t^{1/2}, \tag{5.8}$$

where ϵ_t is the residual of the mean equation and $\hat{h}_t$ its estimated variance.

The diagnostic tests for all five currencies are given in table 14. The runs test and $Q_x(50)$ find no first-order dependence, while $Q_{xx}(50)$ finds second-order dependence only in the DM. On the basis of these two tests, other researchers have concluded that GARCH fits most of these currencies. I have added the Tsay test and the BDS test. The Tsay test finds no evidence of nonlinearity in any of the currencies. The BDS test also finds no evidence of nonlinearity in the CD, JY, and SF, some nonlinearity (at dimensions 8, 9, and 10) for the DM, and strong nonlinearity for the BP.

I must point out that this procedure is often biased in favor of accepting the model. Tauchen (1985) shows that diagnostics of maximum likelihood models may have a different asymptotic distribution from those applied to the raw data and gives examples of diagnostics that are biased toward accepting the model when no adjustments are made to account for the presence of estimated parameters. For the diagnostics in this article, there are standard methods to adjust the asymptotic distribution for the estimated parameters. For $Q_x(50)$, the degrees of freedom are reduced from 50 to $50 - (m + 6)$. For $Q_{xx}(50)$, the degrees of freedom are reduced from *50* to $50 - (q + 6)$. For the goodness of fit, the degrees of freedom are not reduced since there are no parameters to estimate in the standard normal. For the modified Levene test of equality of monthly variances, no adjustment is made because I am not sure how it should be done. In any case, there are so many degrees of freedom (i.e., 119 in the numerator and 2,390 in the denominator) that any adjustment is unlikely to have any important effect.

Table 15 provides evidence that the "nuisance" parameter may affect the asymptotic distribution of the BDS test. I generate GARCH(1, 1) models of the following type:

$$x_t = h_t^{1/2} \epsilon_t, \qquad h_t = 1 + .25\, x_{t-1}^2 + .7\, h_{t-1}, \tag{5.9}$$

where ϵ_t iid $N(0, 1)$. The parameter values of .25 and .7 are close to those estimated for the exchange-rate data. I perform 2,000 replications, each with 1,000 observations. The BDS statistics of GARCH standardized residuals are normally distributed, but their standard errors are substantially below one. Other (unreported) simulations using ARCH with normal errors and GARCH with Student-*t* errors also give similar results. Rather than making the necessary adjustments to the diagnostics, which can be computationally very complicated, I shall merely note their potential biases here.[4]

I also check whether the standardized residuals are normally distributed, which is assumed in the GARCH model above. The Pearson goodness-of-fit test gives $\chi^2(50)$ statistics of 357.73, 97.37, 75.71, 355.29, and 138.96, all significant at the 1% level. Several nonnormal

4. I could adjust the BDS statistic as follows. Ordinarily, I use the critical value of 2.576 from a standard normal distribution to conduct a 1% (two-tailed) asymptotic test. For GARCH standardized residuals, I can multiply 2.576 by the standard deviation reported in table 15. This means the critical values are 1.98, 1.60, 1.42, 1.37, 1.39, 1.47, 1.60, 1.75, and 1.96 for dimensions 2–10 when $(\ell/\sigma) = 1$. Using these critical values, BDS detects no nonlinearity in the SF. It finds some slight nonlinearity in the DM and JY at higher dimensions, some nonlinearity in the CD for dimensions 2–8, and substantial nonlinearity in the BP. This adjustment, however, is only suggestive since it could vary with the sample size as well as the parameters of the model.

TABLE 15 **Distribution of BDS for GARCH(1, 1)-Normal Standardized Residuals (2,000 Replications; 1,000 Points per Replication)**

N	BP	CD	DM	JY	SF
2:					
Mean	−.03	−.02	−.02	−.01	.00
SD	.85	.77	.76	.80	1.00
Skewness	.16	.07	.05	.10	.00
Kurtosis	2.99	3.18	3.18	3.17	3.00
3:					
Mean	−.05	−.03	−.02	−.01	.00
SD	.78	.62	.60	.64	1.00
Skewness	.19	.14	.04	.04	.00
Kurtosis	3.13	3.07	2.98	2.90	3.00
4:					
Mean	−.07	−.04	−.02	.00	.00
SD	.84	.55	.51	.54	1.00
Skewness	.19	.15	.01	.04	.00
Kurtosis	3.27	3.07	2.98	2.86	3.00
5:					
Mean	−.07	−.03	−.01	.00	.00
SD	1.04	.53	.46	.49	1.00
Skewness	.14	.18	.06	.05	.00
Kurtosis	3.31	3.17	3.06	2.92	3.00
6:					
Mean	−.06	−.03	−.01	.00	.00
SD	1.40	.54	.44	.47	1.00
Skewness	.10	.20	.07	.03	.00
Kurtosis	3.35	3.01	2.98	2.96	3.00
7:					
Mean	−.06	−.03	−.01	.00	.00
SD	1.99	.57	.45	.47	1.00
Skewness	.20	.23	.05	.01	.00
Kurtosis	3.22	3.00	3.00	3.03	3.00
8:					
Mean	−.02	−.03	−.01	.00	.00
SD	3.02	.62	.46	.48	1.00
Skewness	.39	.26	.03	.00	.00
Kurtosis	3.23	3.05	2.93	2.97	3.00
9:					
Mean	−.06	−.02	−.01	.00	.00
SD	4.56	.68	.48	.49	1.00
Skewness	.77	.29	.01	−.03	.00
Kurtosis	3.89	3.18	2.93	2.95	3.00
10:					
Mean	−.02	−.01	−.01	.00	.00
SD	7.12	.76	.50	.51	1.00
Skewness	1.60	.33	.01	−.04	.00
Kurtosis	6.90	3.31	2.98	2.95	3.00

NOTE.—Abbreviations are defined in text. GARCH(1, 1) model:

$$x_t = h_t^{1/2}\epsilon_t, \quad h_t = 1 + .25x_{t-1}^2 + .7h_{t-1}, \quad \epsilon_t \text{ iid } N(0,1).$$

distributions are then used, with the Student-*t* and the generalized error distribution appearing to give the best fit.[5] None of the Pearson $\chi^2(50)$ goodness-of-fit tests are significant at the 1% level for these two distributions. Note that the degrees of freedom are reduced by one since there is one parameter to estimate in the Student-*t* and the generalized error distribution. This is a conservative adjustment, as discussed in Kendall and Stuart (1970, vol. 2, ch. 30). The diagnostic tests on the standardized residuals are reported in tables 16 and 17.

The GARCH(1, 1) Student-*t* model is not rejected by any diagnostic test for the CD and the SF. It also seems to fit the DM since the only rejection comes from a BDS statistic of 2.62 at dimension 10. It does not seem to fit the BP and JY. The Tsay test finds nonlinearity for the BP and JY. The BDS test finds nonlinearity in the JY at dimensions greater than 3. The runs test also picks up some dependence in the JY. The $Q_{xx}(50)$ appears to be unusually low for the BP and unusually high for the JY. There may be some problem with the $Q_{xx}(50)$ test and the Tsay test since the Student-*t* has low degrees of freedom for the BP and JY (just below 3), which means that fourth-order moments do not exist. In addition, the standardized residuals for both the BP and JY have enormous coefficients of kurtosis (314.67 and 43.25, repectively.) This is consistent with the estimated models since a Student-*t* distribution with 3 degrees of freedom has infinite kurtosis. However, this is certainly contrary to the spirit of ARCH and GARCH models, particularly in light of the fact that the monthly standardized data in table 13 exhibit substantially lower kurtosis. Furthermore, simulations of GARCH models with the estimated parameters yield data that are so extremely ill behaved that I have no doubt that exchange-rate data could not have been generated by such a model. One possible explanation for these negative results is that the densities of the BP and JY data have very high peaks at zero, which are driving the degrees of freedom of the standardized Student-*t* distribution to low levels.

The GARCH(1, 1)-generalized error distribution also give similar results. It is not rejected by any diagnostic test for the CD and SF. It also seems to fit the DM since the only rejection comes from the $Q_x(50)$ statistic of 61.65, which has a marginal significance level of 0.90%. But it does not seem to fit the BP and JY. The BDS test finds nonlinearity in the BP at dimensions greater than 2, while the Tsay test finds non-

5. The generalized error distribution is used in Nelson (1988). The density function is given by

$$g(x) = .5\ \nu\ \Gamma(3/\nu)\ \Gamma(1/\nu)^{-1.5}\exp(-.5\ |x/\lambda|^{\nu}),$$

where

$$\lambda = \Gamma(1/\nu)^{.5}\ \Gamma(3/\nu)^{-.5}\ 2^{-2/\nu}.$$

When $\nu = 2$, $g(x)$ is the standard normal density. When $\nu = 1$, $g(x)$ is the double exponential density. When $\nu = 0$, $g(x)$ is the uniform density.

TABLE 16 Tests of Nonlinearity: Standardized Residuals, GARCH(1, 1)-Student *t*

N	ℓ	BP	CD	DM	JY	SF
BDS Tests:						
2	1.00	1.79	2.00	−.69	1.77	.26
3	1.00	1.66	1.86	−.43	2.19	.98
4	1.00	1.45	1.70	.13	2.59	1.28
5	1.00	1.03	1.51	.61	2.66*	1.18
6	1.00	.74	1.54	1.21	2.88*	.95
7	1.00	.65	1.50	1.49	2.83*	.56
8	1.00	.37	1.44	1.85	2.88*	.29
9	1.00	.23	1.24	2.22	2.92*	.05
10	1.00	.17	1.04	2.65*	2.92*	−.03
Runs test:		1.86	−1.02	−2.14	−2.57*	−1.54
$N(0,1)$		(.0314)	(.1539)	(.0162)	(.0051)	(.0618)
$Q_x(50)$		47.50	52.00	60.52	39.21	47.73
		(.3320)	(.0757)	(.0115)	(.2476)	(.1337)
$Q_{xx}(50)$		.76	38.75	43.11	389.07*	55.93
		(1.0000)	(.6144)	(.4236)	(.0000)	(.0736)
Tsay test for nonlinearity:						
$M = 2$:		4.52*	.24	.84	24.79*	.53
$F(3,2504)$		(.0039)	(.8684)	(.4743)	(.0000)	(.6659)
$M = 4$:		.96	1.62	.59	2.25	1.16
$F(10,2495)$		(.4765)	(.9477)	(.8234)	(.0130)	(.3133)
$M = 6$:		.55	1.21	.41	1.94*	.88
$F(21,2482)$		(.9505)	(.2308)	(.9916)	(.0064)	(.6188)
$M = 8$:		.62	.82	.59	1.36	.70
$F(36,2465)$		(.9632)	(.7678)	(.9754)	(.0753)	(.9100)
$M = 10$:		.59	.80	.65	1.11	.92
$F(55,2444)$		(.9930)	(.8544)	(.9786)	(.2704)	(.6425)
Skewness		5.14	−.38	−.38	−2.11	.12
Kurtosis		314.67	4.90	11.97	43.25	6.80
Goodness of fit:		69.08	57.35	70.67	56.90	53.59
$\chi^2(50)$		(.0381)	(.2213)	(.0287)	(.2337)	(.3383)

NOTE.—Abbreviations are defined in text. Marginal significance levels are in parentheses.
* Significant at the 1.0% level.

linearity in the JY. The $Q_{xx}(50)$ appears to be unusually small for the BP and quite high for the JY. In addition, the kurtosis is very large for both currencies.

VI. Conclusion

This article shows that daily exchange-rate changes are not independent of past changes. Although there is little linear dependence in the data, the BDS test and autocorrelations of the squared data detect strong nonlinear dependence. Evidence from third-order moments indicates that the nonlinearity is likely to enter through variances rather than through means. This is consistent with the presence of conditional heteroskedasticity.

Generalized ARCH models with normal and nonnormal conditional

TABLE 17 Tests of Nonlinearity: Standardized Residuals, GARCH(1, 1)-Generalized Error Distribution

N	ℓ	BP	CD	DM	JY	SF
BDS Tests:						
2	1.00	2.32	2.10	−1.04	1.23	.00
3	1.00	2.76*	1.93	−.86	1.44	.58
4	1.00	2.83*	1.73	−.24	1.84	.82
5	1.00	2.77*	1.54	.27	1.92	.69
6	1.00	2.87*	1.58	.91	2.22	.44
7	1.00	3.21*	1.57	1.24	2.21	.06
8	1.00	3.46*	1.56	1.66	2.33	−.16
9	1.00	3.93*	1.40	2.06	2.42	−.36
10	1.00	4.43*	1.18	2.52	2.48	−.41
Runs test:		1.78	−1.26	−1.98	−2.18	−1.94
$N(0,1)$		(.0375)	(.1038)	(.0239)	(.0146)	(.0262)
$Q_x(50)$		69.68*	53.32	61.65*	51.23	51.98
		(.0081)	(.0630)	(.0090)	(.0292)	(.0649)
$Q_{xx}(50)$		13.14	39.84	59.05	84.37*	50.19
		(1.0000)	(.5662)	(.0422)	(.0001)	(.1806)
Tsay test for nonlinearity:						
$M = 2$		.80	.22	.56	7.52*	.54
$F(3,2504)$		(.4066)	(.8818)	(.6457)	(.0001)	(.6592)
$M = 4$		.99	1.60	.50	1.80	1.11
$F(10,2495)$		(.4497)	(.1004)	(.8910)	(.0556)	(.3503)
$M = 6$		.72	1.20	.43	1.39	.88
$F(21,2482)$		(.8168)	(.2396)	(.9886)	(.1106)	(.6188)
$M = 8$		.85	.81	.61	1.01	.72
$F(36,2465)$		(.7220)	(.7823)	(.9677)	(.4527)	(.8915)
$M = 10$		.84	.79	.68	.88	.92
$F(55,2444)$		(.7933)	(.8678)	(.9656)	(.7217)	(.6425)
Skewness		1.44	−.06	−.18	−1.31	.17
Kurtosis		36.98	4.86	8.94	30.04	6.35
Goodness of fit:		73.11	53.57	64.05	73.36	55.34
$\chi^2(50)$		(.0182)	(.3390)	(.0874)	(.0173)	(.2802)

NOTE.—Abbreviations are defined in text. Marginal significance levels are in parentheses.
* Significant at the 1.0% level.

distributions are estimated to try to account for conditional heteroskedasticity. Conditional normality is strongly rejected in all currencies in favor of conditional nonnormal distributions. In fact, the GARCH(1, 1) model using either the Student-t or generalized error distribution can describe the CD and SF very well and the DM reasonably well. None of the GARCH models, however, can fit the BP and JY satisfactorily. Regardless of the fit of the model, diagnostics show that the GARCH(1, 1) model can account for most of the nonlinearity in the data.[6]

Whether I am able to successfully model all five currencies, the

6. It is interesting to note that Scheinkman and LeBaron (in this issue) find important nonlinearities in stock returns after allowing for conditional heteroskedasticity. Private correspondence with William Brock and Blake LeBaron indicates that stock returns exhibit many statistically significant third-order moments. This may explain why ARCH and GARCH do not capture all the nonlinearity in stock returns.

central message remains as follows: conditional heteroskedasticity accounts for a large part of the nonlinearity in daily exchange rates. Thus, models of short-term exchange-rate determination should be developed to explain this stylized fact.

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