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The Variance Gamma (V.G.) Model for Share Market Returns*

A new stochastic process, termed the V.G. (Variance Gamma) process, is proposed as a model for the uncertainty underlying security prices. The unit period distribution is normal conditional on a variance that is distributed as a gamma variate. Its advantages include long tailedness, continuous-time specification, finite moments of all orders, elliptical multivariate unit period distributions, and good empirical fit. The process is pure jump, approximable by a compound Poisson process with high jump frequency and low jump magnitudes. Applications to option pricing show differential effects for options on the money, compared to, in, or out of the money.

I. Introduction

The purpose of this article is to introduce a continuous-time stochastic process, termed the V.G. model, for modeling the underlying uncertainty driving stock market returns. The broad objective is to provide a practical and empirically relevant alternative to the role of Brownian motion, as the martingale component of the motion in log prices. The emphasis on proposing a stochastic process, as opposed to just a distribution for unit period returns, is important for applications to European call option pricing that do not merely compute risk-neutral expectations but account for risk aversion via the identification of an explicit change of measure (Harrison and Pliska 1983).

The practical and empirically relevant properties sought in the proposed process include (1) long tailedness relative to the normal for daily returns, with returns over longer periods approaching normality (Fama 1965); (2) finite moments for at least the lower powers of returns; (3) consistency with an underlying, continuous-time stochastic process, with independent stationary increments, and with the distribution of any

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increment belonging to the same simple family of distributions irrespective of the length of time to which the increment corresponds (thereby permitting sampling and analysis through time in a straightforward fashion); and (4) extension to multivariate processes with elliptical multivariate distributions that thereby maintain validity of the capital asset pricing model (Owen and Rabinovitch 1983).

The literature on market returns includes a number of models. In addition to Brownian motion and the normal distribution, Mandelbrot (1963) put forward the symmetric stable distribution; Press (1967) introduced a compound events model combining normally distributed jumps at Poisson jump times; and Praetz (1972) suggested the t distribution. More recently, Bookstaber and McDonald (1987) have proposed a generalized beta distribution.

Brownian motion fails property 1. The symmetric stable fails on properties 2 and 3. The Praetz t distribution fails on property 3 as it is not possible to construct a stochastic process with the property 3 and distributions of any increment being a t distribution irrespective of length of time interval considered since the sum of independent t-variables is not a t-variable (cf. Blattberg and Gonedes 1974). We are not aware of the stochastic process satisfying property 3 that underlies the generalized beta. Though the compound events model of Press possesses all the four properties described above, the proposed V.G. model has a further advantage in being a pure jump process of, in the main, a large number of small jumps. In fact, we show that the V.G. model is a limit of a particular sequence of compound events models in which the arrival rate of jumps approaches infinity, while the magnitudes of the jumps are progressively concentrated near the origin. In this sense, the V.G. model respects the intuition underlying the sample path continuity of Brownian motion as a model.

Section II concentrates on the V.G. model as a model for the unit period return distribution, including its elliptical multivariate generalization. Section III takes up the stochastic process properties. Section IV presents an application to risk-neutral option pricing and Section V concludes.

II. The V.G. Model

A. The V.G. Model Unit Period Distribution

The formulation of the V.G. model is comparable to that of Praetz (1972) in that it is obtained from the normal by mixing on the variance parameter. Formally, let R_t be the return over a unit time period, say $R_t = (S_{t+1}/S_t)$, where S_t is the price of the stock at time t. Suppose that $\log (R_t)$ is normally distributed with mean μ and a random variance $\sigma^2 V$, where μ and σ^2 are known constants. The distribution of V is taken to be gamma, with parameters c, γ, and density $g(v)$ given by

$$g(v) = \frac{c^{\gamma}v^{\gamma-1}e^{-cv}}{\Gamma(\gamma)},$$

where Γ is the gamma function. If $X = \log(R) - \mu$ (dropping the t subscript for notational convenience), then the density of X, $f(x)$ is

$$f(x) = \int_0^{\infty} [e^{-x^2/(2\sigma^2 v)}/(\sigma\sqrt{2\pi v})]\, g(v)dv, \tag{1}$$

and there is no closed-form expression for f. However, the characteristic function for X, $\phi_X(u)$, has a closed-form expression obtained easily by conditioning on V,

$$\phi_X(u) = [1 + (\sigma^2\upsilon/m)(u^2/2)]^{-m^2/\upsilon},$$

where $m = \gamma/c$ is the mean of the gamma density $g(v)$ and $\upsilon = \gamma/c^2$ is its variance. It is clear from the form of this characteristic function that just $\sigma^2\upsilon/m$ and m^2/υ are identified. Since σ^2 serves as a scale parameter for V, we take the mean of V to be unity: $m = 1$ or $\gamma = c$.

It is shown below that the variable V can be viewed as a random time change and this setting of m is consistent with supposing that the expected random time change is unity for the unit period return. The characteristic function of the unit period return distribution therefore is

$$\phi_X(u) = [1 + \sigma^2\upsilon u^2/2]^{-1/\upsilon}. \tag{2}$$

B. Moments of the V.G. Distribution and Parameter Interpretation

The higher moments of the V.G. are obtained by conditioning on V. The conditional expectation of X^n being $a_n V^{n/2}\sigma^n$, $a_n = (n-1)(n-3)\ldots(1)$ for n even and 0 for n odd. The expectation of V^k is $\Gamma(\gamma + k)/(\Gamma(\gamma)c^k)$ or $\upsilon^k\Gamma(\upsilon^{-1} + k)/\Gamma(\upsilon^{-1})$. Hence the V.G. has finite moments of all orders and in particular the second and fourth moments are given by $EX^2 = \sigma^2$, and $EX^4 = 3\sigma^4(1 + \upsilon)$.

The kurtosis is therefore $3(1 + \upsilon)$. Since the kurtosis under normality is 3, the proportional excess of the kurtosis over 3 is υ; hence υ may be regarded as a measure of the degree of long tailedness. Though this is less satisfactory from a pure long-tailedness viewpoint than altering the rate of tail decay (which, for the V.G., is still exponential), it does represent increased tail probability. Furthermore, if moments of all orders are required, the rate of tail decay can not be easily altered.

C. Effects of Varying Kurtosis

Figure 1 below illustrates the effects of varying ν on the density function of the unit period V.G. distribution. The value of σ for the construction of figure 1 was 0.4 and υ was set at 0.25 and 1.0, the two

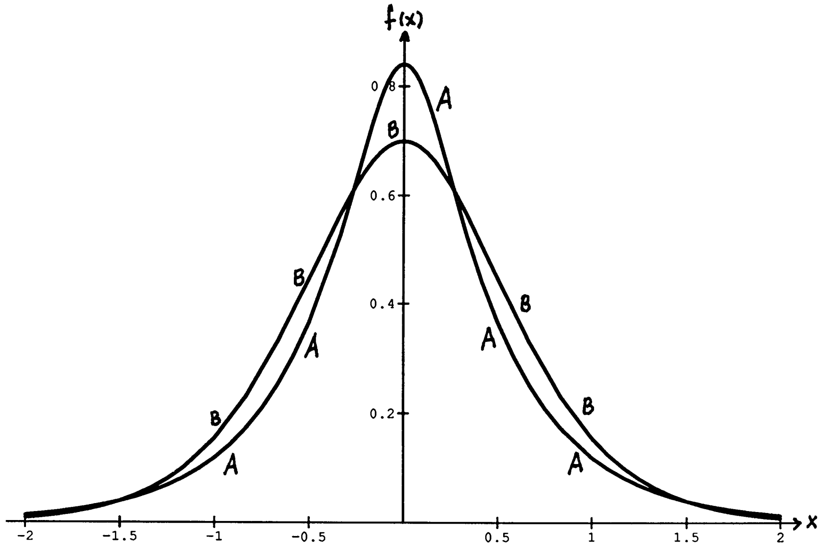

FIG. 1.—Effect of a change in ν from 1.0 to 0.25 (σ = 0.4). Curve A has ν = 1.0; curve B has ν = 0.25.

extreme values used in tables 1 and 2 of Section IV. The effect of raising ν is to increase probability near the origin, as well as to increase tail probabilities at the expense of probability in the intermediate range, which, for the example of figure 1, is from around .4 to 1.5 on the horizontal axis, independent of sign.

D. *Relationship of V.G. to Praetz* t

Praetz took $1/V$ to be distributed as a gamma random variable which results in a t distribution, as already mentioned. Though Grosswald (1976) has shown that the t distribution is infinitely divisible, hence consistent with a process of stationary independent increments, the analytical structure of the underlying process is quite complex and difficult to work with. In particular, the densities do not belong to the same simple family for increments over intervals of arbitrary length. Conversely, our choice of the gamma density employs the process of independent gamma increments, whose structure is well known in the probability literature (e.g., Ferguson and Klass 1972), to establish consistency of unit period returns with an easily described underlying continuous-time stochastic process.

E. *Estimation by Moment Methods*

The parameters σ and υ may be estimated by moment methods. Employing the data X_i, $i = 1, \ldots, N$, one obtains an unbiased and consistent estimate of σ^2 by

$$\hat{\sigma}^2 = \sum_i X_i^2/N$$

and a consistent estimate of υ by

$$\hat{\upsilon} = \left\{\sum_i X_i^4/N\right\}(3\hat{\sigma}^4)^{-1} - 1.$$

F. Maximum-Likelihood Estimation of Parameters

Direct maximum-likelihood estimation using (1) is likely to prove computationally expensive (empirical characteristic function methods are also not always successful; see Epps and Pulley 1985; Madan and Seneta 1987*a*), but the transformed maximum-likelihood methods using (2) of Madan and Seneta (1987*b*, 1989) may be used. As shown by Madan and Seneta (1989), the density of $\Theta = uX(\text{mod}2\pi)$, $h(\theta)$ (when this is a continuous function of bounded variation on $[-\pi, \pi]$) may be written as

$$h(\theta) = 1/(2\pi) + \pi^{-1}\sum_{k=1}^{\infty}\phi_X(ku)\cos(k\theta). \tag{3}$$

Madan and Seneta (1987*b*) implement the transformed maximum-likelihood estimation procedure for data on stock market returns.

G. Empirical Relevance of the V.G. Model for Stock Market Returns

Empirically, the V.G. model is a good contender as a model for describing daily stock market returns, as reported in Madan and Seneta (1987*b*). The V.G. model (in Madan and Seneta 1987*b*) was compared with the normal, the stable, and the Press compound events model (termed ncp), using a chi-squared goodness-of-fit statistic on seven class intervals for unit sample variance data on 19 stocks quoted on the Sydney Stock Exchange. The class intervals used were $-\infty$, -1, -0.75, -0.25, 0.25, 0.75, 1.0, ∞. Accordingly the chi-squared statistics had 6 degrees of freedom. For 12 of the 19 stocks studied, minimum chi squared was attained by the V.G. model, when compared to the ncp, stable, and normal processes. The remaining seven cases were best characterized by the ncp for five cases, the stable for two cases, and none for the normal distribution.

H. Power Series Representation for Density

From the characteristic function for the V.G. model it may also be observed that X can be written as $Y - Z$ where Y, Z are independently and identically distributed (i.i.d.) gamma random variables with mean $\sigma/\sqrt{2\nu}$ and variance $\sigma^2/2$. This follows on factoring $(1 + a^2u^2)$ as $(1 - iau)(1 + iau)$ for $a^2 = \sigma^2\upsilon/2$. This representation provides an interesting decomposition of the process of stock price movements into independent processes for the increases and the decreases (see Sec. III). It also allows us to obtain a power series representation for $f(x)$, using the results of Kullback (1936),

$$f(x) = \frac{\sqrt{2/\upsilon}}{\sigma} \frac{(x\sqrt{2/\upsilon}/\sigma)^{(2/\upsilon - 1)/2}}{2^{(2/\upsilon - 1)/2}\Gamma(1/\upsilon)\sqrt{\pi}} K_{(2/\upsilon - 1)/2}(x\sqrt{2/\upsilon}/\sigma), \tag{4}$$

where $K_w(x)$ is a Bessel function of the second kind of order w and of imaginary argument (see Whittaker and Watson 1962). It is known (Teichroew 1957) that there is a closed form for the density if $(1/\upsilon)$ is an integer, but since υ is an unknown parameter, this is not an advantage.

I. Multivariate Extension for the V.G.

A multivariate extension of the V.G. model is easily obtained by letting X now denote a vector of random variables distributed conditional on the nonnegative random variable V as a multivariate normal with mean vector zero and variance-covariance matrix ΣV. Again, by conditioning on V, the joint characteristic function of X, $\phi_X(u)$, where u is now an appropriately dimensioned vector, is easily seen to be

$$\phi_X(u) = (1 + \upsilon u^T \Sigma u/2)^{-1/\upsilon}, \tag{5}$$

which generalizes (2). Since ϕ_X is a function of u via the quadratic form $u^T\Sigma u$, the joint density is elliptical, and, as Kelker (1970, theorem 6) shows, the conditional expectation function of X_i, given the other X's, is linear in these X's. In fact, as conditional on V, we have multivariate normality, the slope coefficients of the linear conditional expectation are easily seen to be independent of V. This linearity of conditional expectations is useful for conducting tests of the capital asset pricing model, the validity of which, in the elliptical context, was demonstrated by Owen and Rabinovitch (1983). Test procedures along the lines of Gibbons (1982) could be implemented.

A shortcoming of the multivariate model is that υ is the same for all the marginal distributions, hence they have identical kurtosis.

Estimation in the multivariate context may be done by estimating Σ, using moment methods, and estimates of υ being obtained from transformed maximum likelihood applied to the univariate series $\Theta_i = uX_i(\text{mod}2\pi)$, using (3) for the likelihood. Alternatively, multivariate Fourier methods may be employed to approximate the joint density, $h(\theta)$ of $\Theta_i = uX_i(\text{mod}2\pi)$, by

$$h(\theta) = (1/2\pi)^m + \sum_{\alpha=1}^{A} \sum_{j=1}^{J} 2/(2\pi)^m \, \phi_X(juk_\alpha)\cos(juk_\alpha^T\theta), \tag{6}$$

where the summation on α runs over a set of multi-indices k_α (Edmunds and Moscatelli 1977).

III. The V.G. Stochastic Process

A. The V.G. Stochastic Process and Random Time Change

The continuous-time stochastic process $Y(t)$, which is consistent with the V.G. model as the distribution for the unit period motion $Y(t + 1) - Y(t)$, is given by Brownian applied to random time change,

$$Y(t) = b(G(t)), \tag{7}$$

where $G(t)$ is the process of i.i.d. gamma increments with mean τ and variance $\upsilon\tau$ over intervals of length τ, and $b(t)$ is an independent Brownian motion of zero drift and variance rate σ^2.

Consistent with the multivariate extension of the V.G. in Section II*I* above, one could take the Brownian motion in (7) to be multivariate with zero drift and covariance rates given by Σ. This has the interesting economic interpretation of supposing that economically relevant time is random in that a calendar year is, in economic terms, sometimes less than one economic year and sometimes more, $G(t)$ being the number of economic years in t calendar years. The random time $G(t)$ is here an economywide entity, much like systematic risk, and it may be thought of as a measure of systematic time. More informally, one may think of $G(t)$ as a formal statement of the remark, ''Didn't have much of a year this year,'' by allowing for an interpretation of how much of a year one actually had. A suggestive heuristic candidate for $G(t)$ could be the cumulated gross domestic product. (Primarily for notational convenience we restrict attention in the rest of this section to the univariate case.)

B. The V.G. Stochastic Process as Gains and Losses

Another representation for the V.G. process is obtained on exploiting the observation that the V.G. is the difference of two i.i.d. gamma variates, in which case one may also write

$$Y(t) = U(t) - W(t), \tag{8}$$

where $U(t)$, $W(t)$ are processes of independent gamma increments with means $(\sigma/\sqrt{2\upsilon})h$ and variances $\sigma^2 h/2$ for increments over intervals of length h.

In this representation there is a decomposition of Y into a gains process $U(t)$ and a process of losses $W(t)$, the gains and losses being independent and having the same mean and variance rates.

C. The Distribution of the Short and Long Motions

It may be observed from the characteristic function, $\phi_{A(t)}(u)$, of the standardized variate $A(t) = Y(t)/(\sigma\sqrt{t})$,

$$\phi_{A(t)}(u) = [1 + \nu u^2/(2t)]^{-t/\nu},$$

that the distribution of $Y(t)$ for large t approaches normality.

A comparison of the characteristic function for $A(t)$ with (2) reveals that the kurtosis for $A(t)$, as derived in Section II*B*, is $3(1 + \nu/t)$. Hence, for large t we have the kurtosis of the normal. This property of the V.G. is consistent with empirical evidence on stock returns, where long tailedness is present for daily returns, but monthly returns tend to be normally distributed.

D. Compound Poisson Approximations for the V.G. Process

All of the processes $b(t)$, $G(t)$, $Y(t)$, $U(t)$, and $W(t)$ are examples of Lévy processes, a natural continuous-time analog of a sequence of partial sums of independently and identically distributed random variables. A detailed survey of such processes in general can be found in Fristedt (1974, pp. 241–396). It is clear from (7) and (8) that central to the properties of our V.G. process are processes with stationary independent gamma increments since all of $G(t)$, $U(t)$, and $W(t)$ are of this form.

THEOREM 1. For a (gamma) process $Z(t)$ of independent stationary gamma increments having mean μ and variance τ^2 for unit time, the Lévy representation of the distribution of increments per unit time has no Gaussian component, and the Lévy measure is given by $\rho(dz) = 0$ for $z \le 0$; $= (\mu/\theta)(\exp - z/\theta)/z\, dz$ for $z > 0$, where $\theta = \tau^2/\mu$. The process $Z(t)$ is the limit of approximation as $n \to \infty$ of a compound Poisson process with arrival rate $\mu\beta_n/\theta$ and i.i.d. jumps of size described by the density $(e^{-z/\theta}/z\beta_n)I_{\{z>1/n\}}$, where $\beta_n = \int_{1/n}(e^{-z/\theta}/z)\, dz$ and $I_{\{z>1/n\}}$ is the indicator function of the set $\{z|z > 1/n\}$.

Proof. See the Appendix. Q.E.D.

This result also shows that the nondecreasing process $Z(t)$ is pure jump and so also is $G(t)$. Theorem 1 identifies the compound Poisson processes that approximate processes like $Z(t)$, $G(t)$, $U(t)$, and $W(t)$. The nature of the approximating compound Poisson process is to equate to zero all jumps of the process $Z(t)$ of size $\le 1/n$. While the number of jumps of $Z(t)$ in any finite interval of length h is, at most, countable, it is clear from the nature of ρ that there are infinitely many jumps of vanishing size in any interval. As n increases, the arrival rate tends to infinity while the jump magnitudes tend to be concentrated near the origin.

The approximation to the V.G. process is a corollary to this theorem.

COROLLARY TO THEOREM 1. The V.G. process $Y(t)$ is a pure jump process that can be approximated as the difference of two independent compound Poisson processes, each of which has arrival rate $(1/\nu)\beta_n$ and i.i.d. jumps described by the density of theorem 1 where $\theta = \sigma\sqrt{\upsilon/2}$. Furthermore, the Lévy measure for the V.G. process is

$$F(dx) = e^{-(|x|/\sigma)(\sqrt{2/\upsilon})}/(\upsilon|x|)dx, \quad x \neq 0 \text{ and } F(\{0\}) = 0.$$

Proof. This follows from (8) and theorem 1, by putting "μ" $= (\sigma/\sqrt{2\upsilon})$ and "τ^2" $= \sigma^2/2$. Q.E.D.

The compound Poisson approximation to the V.G. given by the above corollary is most useful in applications that employ Monte Carlo methods to simulate asset price paths. Such applications include the valuation of complex derivative securities like the call and put option components of mortgages (Hendershott and Van Order 1987). Parameter estimation in such models can involve the use of the recently proposed simulation moment estimator (Duffie and Singleton 1989).

E. *The Fine Structure of the Jumps*

A process of independent stationary gamma increments, $Z(t)$, over $t \in [0, 1]$ (and so correspondingly over any specific unit time interval), has, furthermore, a representation in terms of its jumps, and the joint distribution of the jump sizes is available (see Ferguson and Klass [1972] and, more concisely, Ferguson [1973], pp. 218–19).

THEOREM 2. For any process $Z(t)$ of independent stationary gamma increments having mean μ and variance τ^2 for unit time (with $\theta = \tau^2/\mu$), for $t \in [0, 1]$,

$$Z(t) = \sum_{j=1}^{\infty} J_j I_{[0,t)}(U_j),$$

with almost sure convergence, where J_j is the size of the jth largest jump in [0,1] and the U_j, $j \geq 1$, are i.i.d. random variables uniformly distributed on [0,1], and independent of J_j, $j \geq 1$.

The joint distribution of the first k jumps, starting with the largest, is given by the following recursive scheme. Let

$$N(x) = -(\mu/\theta)\int_{x/\theta}^{\infty} e^{-w}/w \, dw.$$

Then

$$P(J_1 \leq x_1) = \exp(N[x_1])$$

for $x_1 > 0$; and $j = 2, 3, \ldots,$

$$P(J_j \leq x_j | J_{j-1} = x_{j-1}, \ldots, J_1 = x_1) = \exp(N[x_j] - N[x_{j-1}])$$

for $0 < x_j < x_{j-1}$.

Proof. This follows with appropriate substitutions and simplifications from Ferguson and Klass (1972). Q.E.D.

In particular, the distribution function of the kth largest jump of $Z(t)$ is

$$\Psi^k(x) = \exp(N[x]) \sum_{j=0}^{k-1} (-1)^j N(x)^j/j!$$

for $x > 0$.

We note that a property of the V.G. process $Y(t)$ is that there are infinitely many jumps in any interval, and indeed the characteristic of jumps vanishing in size is shared by the process of independent stable increments studied by McCulloch (1978). Given that stock prices are constrained to move in multiples of one cent, such a description is more realistic than a description by a diffusion process.

The level of detail about the jump structure provided by theorem 2 could be useful in evaluating securities that refer specifically to these jump magnitudes. For example, if the price path is taken as piecewise constant from a transaction tape, then contracts may well be written and marketed that protect investors against big jumps. This could be done, for example, by contracting a currency swap to neutralize the effect of big jumps. The valuation of such a contract would require a knowledge of the fine structure of jumps as provided by theorem 2.

IV. Risk-neutral Option Pricing

A. *Risk Aversion and V.G. Option Pricing*

Suppose we model the underlying uncertainties driving the security price by the V.G. process $Y(t)$ with parameters σ and υ. If we let the mean corrected unit period log difference of the stock price have the V.G. distribution of Section II, then the motion of log $S(t)$ is given by

$$\log S(t) = \mu t + Y(t),$$

where μ is the drift on the log price.[1]

The absence of arbitrage opportunities implies that the price of a European call option on the stock W with exercise price E and maturity T in a world with a fixed interest rate of r is given by the discounted expected value of the option price at maturity, where the expectation is taken under an absolutely continuous change of measure (Harrison and Pliska 1981). Option pricing in markets with risk-averse investors therefore requires the identification of the change of measure. For processes with jump discontinuities, this option price will depend on the stock's drift rate as observed by Madan, Milne, and Shefrin (1989). Option pricing with the V.G. process, taking account of the change of measure, is the subject matter of another paper (Madan and Milne 1989).

1. The drift for the stock price is the expected unit period return, $\lambda = E(S(1) - S(0))/S(0)$, and for the V.G. this is related to μ by $\mu = \lambda + 1/\upsilon \ln(1 - \upsilon\sigma^2/2)$.

B. Risk-neutral V.G. Option Pricing

Here we restrict attention to the risk-neutral case and comment on the effect of the kurtosis parameter ν. The resulting formula is also relevant to the case in which the density process for the change of measure has zero covariation with the stock price process. In this case the stock price drift (see n. 2 below) must equal the interest rate.

For this special case one may write,

$$W = e^{-rT} \int_{E}^{\infty} (S_T - E) f(S_T) dS_T, \tag{9}$$

where $f(S_T)$ is the density of the stock price at maturity. The option price may be easily evaluated by conditioning on V, the random time change component of the V.G. process. Given V, we have a Black-Scholes situation and the option price is obtained using the Black-Scholes formula. It remains to integrate out the conditioning variable V with respect to its gamma density, and this is accomplished by numerical integration procedures.

C. Risk-neutral V.G. Option Prices and Kurtosis

The option price (9) has one extra parameter over Black-Scholes and this is υ, the percentage excess of the kurtosis over the normal kurtosis. We report on the impact of this parameter in table 1. Since the V.G. has greater kurtosis at lower maturities, we investigate the cases of both a 1-month maturity, or $T = .08333$, and a 3-month maturity, or $T = .25$. The interest rate is taken at 10%, the exercise price is 100, and σ^2 is 0.40.[2]

Table 1 shows the option prices given by the formula (9) for various stock prices and levels of kurtosis, with $T = .08333$.

One observes from table 1 that the effect of increasing the kurtosis is to initially lower the option price, but as the kurtosis is further increased, the V.G. option price rises above the Black-Scholes value. For both in-the-money and out-of-the-money options, the option price did not fall as swiftly and rose above Black-Scholes sooner, when kurtosis was increased, than was the case for on-the-money options.

In table 2, the maturity is increased to $T = .25$, other arguments being as in table 1.

As may be observed by comparing tables 1 and 2, the lower kurtosis for the larger maturity requires a higher υ parameter for the V.G. option price to come up to the Black-Scholes value.[3] The general pattern that is observed in table 1—of the initial drop in the V.G. value being greater and the rise slower as we raise υ for on-the-money op-

2. The drift λ equals 10%, and μ is calculated in accordance with the formula in n. 1 above.

3. Note that the kurtosis for the distribution at maturity t is υ/t (see Sec. IIIC).

TABLE 1 Black-Scholes and Risk-neutral V.G. Option Values for Varying Kurtosis and Stock Price

Stock Price	Black-Scholes	% Change in Kurtosis 25	50	75	100
90	3.24	3.23	4.12	5.16	6.24
92	3.94	3.71	4.60	5.68	6.81
94	4.74	4.25	5.15	6.26	7.43
96	5.62	4.89	5.78	6.93	8.14
98	6.60	5.62	6.52	7.71	8.96
100	7.67	6.52	7.43	8.64	9.92
102	8.82	7.66	8.58	9.81	11.11
104	10.05	9.02	9.96	11.20	12.51
106	11.37	10.51	11.48	12.73	14.05
108	12.75	12.09	13.10	14.36	15.68
110	14.21	13.75	14.78	16.05	17.38

TABLE 2 Black-Scholes and Risk-neutral V.G. Option Values for Varying Kurtosis and Stock Price

Stock Price	Black-Scholes	% Change in Kurtosis 25	50	75	100
90	8.39	7.80	7.77	8.23	8.96
92	9.34	8.61	8.49	8.92	9.64
94	10.35	9.47	9.26	9.66	10.38
96	11.41	10.41	10.11	10.47	11.18
98	12.52	11.42	11.02	11.35	12.05
100	13.68	12.50	12.02	12.31	13.01
102	14.89	13.65	13.12	13.39	14.09
104	16.15	14.89	14.35	14.61	15.31
106	17.46	16.20	15.69	15.97	16.68
108	18.80	17.58	17.12	17.43	18.15
110	20.19	19.02	18.62	18.96	19.71

tions as opposed to in- or out-of-the-money options—is maintained in table 2.

V. Conclusion

A new stochastic process for the underlying uncertainty driving security prices was proposed. The process was termed the V.G., for its distribution is normal conditional on a variance that is distributed as a gamma variate. The new process was shown to be long tailed relative to the normal for a motion over smaller time intervals and approached normality for motions over longer intervals of time. The stochastic process is pure jump and approximable by a particular compound Poisson process with high jump frequency and jump magnitude concentrated near the origin. Furthermore, the joint distributions of the jumps in the order of magnitude were identified. It was also observed that the unit period distributions possessed finite moments of all orders and have good empirical fit (Madan and Seneta 1987*a*). In addition, a generalization to a multivariate stochastic process was made that had elliptical unit period distributions consistent with the capital asset pricing model.

Applications to option pricing were also made, and a differential effect was observed on options on the money, as opposed to those that are in or out of the money. Increases in kurtosis have a greater upward effect on options that are either in or out of the money as opposed to those that are on the money.

Appendix

Proof of Theorem 1

The log characteristic function $\ln\phi(u)$ of any process with independent and stationary increments (such as the process $Z(t)$) can be written uniquely in the Lévy form:

$$\ln\phi(u) = t[ibu - \frac{1}{2}c^2u^2 + \int (e^{iuz} - 1 - iuzI(\{|z|\leq 1\}))K(dz)], \quad \text{(A1)}$$

where $-\infty < b < \infty$, and K is a positive measure on the real line satisfying $K(\{0\}) = 0$ and $\int(x^2 \wedge 1)K(dx) < \infty$ (Jacod and Shiryaev 1987, pp. 107, 76).

For the process $Z(t)$, $\ln\phi(u) = -t(\mu/\tau)^2\ln(1 - iu\theta)$ and (A1) are easily checked directly by power series expansion for $K(dz) = 0$ for $z \leq 0$; $= \rho(dz)$, $z > 0$; while $c^2 = 0$ and $b = \mu(1 - e^{-\mu/\tau^2})$. The integrability of $(z^2 \wedge 1)\rho(dz)$ follows on inspection. Z(t) is therefore pure jump with jump compensator $dt\rho(dz)$.

For the approximation, define $\rho_n(dz)$ by $\rho(dz)I_{\{z>1/n\}}$ and $F_n(dz) = \rho_n(dz)/(\mu\beta_n/\theta)$ and consider the log characteristic function defined by

$$\ln\phi_n(u) = t(\mu\beta_n/\theta)[iu \int_0^\infty zI_{\{z\leq 1\}}\, F_n(dz) + \int_0^\infty (e^{iuz} - 1 - iuzI_{\{z\leq 1\}})F_n(dz)]. \quad \text{(A2)}$$

The measures $(\mu\beta_n/\theta)F_n$ converge to ρ by construction, and ϕ_n, the characteristic function for the required compound Poisson process, converges to ϕ. This implies convergence of the compound Poisson processes to the process $Z(t)$.

References

Bookstaber, Richard M., and McDonald, James B. 1987. A general distribution for describing security price returns. *Journal of Business* 60 (July): 401–24.

Blattberg, Robert C., and Gonedes, Nicholas. 1974. A comparison of the stable and student distributions as statistical models for stock prices. *Journal of Business* 47 (April): 244–80.

Duffie, D., and Singleton, K. 1989. Simulated moments estimation of Markov models of asset prices. Research Paper no. 1083. Stanford, Calif.: Stanford University, Department of Finance.

Edmunds, D. E., and Moscatelli, V. B. 1977. Fourier approximation and embeddings of Sobolev spaces. *Dissertationes Mathematicae*. Vol. 145. Warsaw: Polska Akademia Nauk, Insstytut Matematyczny.

Epps, T. W., and Pulley, L. B. 1985. Parameter estimates and tests of fit for infinite mixture distributions. *Communications in Statistics—Theory and Method* 14, no. 12:3125–45.

Fama, E. 1965. The behaviour of stock market prices. *Journal of Business* 38 (January): 34–105.

Ferguson, T. S. 1973. A Bayesian analysis of some non-parametric problems. *Annals of Statistics* 1:209–30.

Ferguson, T. S., and Klass, M. J. 1972. A representation of independent increment processes without Gaussian components. *Annals of Mathematical Statistics* 43:1634–43.

Fristedt, B. 1974. Sample properties of stochastic processes with stationary, independent increments. In P. Ney and S. C. Port (eds.), *Advances in Probability and Related Topics*. Vol. 3. New York: Marcel Dekker.

Gibbons, Michael R. 1982. Multivariate tests of financial models: A new approach. *Journal of Financial Economics* 10 (March): 3–27.

Grosswald, E. 1976. The student *t*-distribution of any degree of freedom is infinitely divisible. *Zeitschrift fuer Wahrscheinlichkeitstheorie verw. Gebeite* 36:103–9.

Harrison J. M., and Pliska, S. R. 1983. A stochastic calculus model of continuous trading: Complete markets. *Stochastic Processes and Their Applications* 15:313–16.

Hendershott, P. H., and Van Order, R. 1987. Pricing mortgages: An interpretation of the models and results. *Journal of Financial Services Research* 1 (September):19–56.

Jacod, J., and Shiryaev, A. N. 1987. *Limit Theorems for Stochastic Processes*. Berlin: Springer-Verlag.

Kelker, D. 1970. Distribution theory of spherical distributions and a location scale parameter generalization. *Sankhya A* 32:419–30.

Kullback, S. 1936. The distribution laws of the difference and quotient of variables independently distributed in Pearson type III laws. *Annals of Mathematical Statistics* 7:51–53.

McCulloch, J. H. 1978. Continuous time processes with stable increments. *Journal of Business* 51 (October): 601–19.

Madan, D. B., and Milne, F. 1989. Option pricing with V.G. martingale components. Unpublished manuscript. College Park: University of Maryland, Department of Finance.

Madan, D. B., Milne, F., and Shefrin, H. 1989. The multinomial option pricing model and its Brownian and Poisson limits. *Review of Financial Studies* 2 (Summer): 251–66.

Madan, D. B., and Seneta, E. 1987*a*. Simulation of estimates using the empirical characteristic function. *International Statistical Review* 55:153–61.

Madan, D. B., and Seneta, E. 1987*b*. Chebyshev polynomial approximations and characteristic function estimation. *Journal of the Royal Statistical Society*, ser. B, 49:163–69.

Madan, D. B., and Seneta, E. 1989. Characteristic function estimation using maximum likelihood on transformed variables. *Journal of the Royal Statistical Society*, ser. B, 51:281–85.

Mandelbrot, B. B. 1963. New methods in statistical economics. *Journal of Political Economy* 71:421–40.

Owen, J., and Rabinovitch, R. 1983. On the class of elliptical distributions and their applications to the theory of portfolio choice. *Journal of Finance* 38 (June): 745–52.

Praetz, P. D. 1972. The distribution of share price changes. *Journal of Business* 45 (January): 49–55.

Press, S. J. 1967. A compound events model for security prices. *Journal of Business* 40 (July): 317–35.

Teichroew, D. 1957. The mixture of normal distributions with different variances. *Annals of Mathematical Statistics* 28:510–12.

Whittaker, E. T., and Watson, G. N. 1962. *Modern Analysis*. Cambridge: Cambridge University Press.