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The Asymptotic Distribution of Extreme Stock Market Returns*

This article presents a study of extreme stock market price movements. According to extreme value theory, the form of the distribution of extreme returns is precisely known and independent of the process generating returns. Using data for an index of the most traded stocks on the New York Stock Exchange for the period 1885–1990, I show empirically that the extreme returns obey a Fréchet distribution.

I. Introduction

Extreme price movements like stock market booms and crashes are some of the most puzzling phenomena in finance. These events, which are of great importance for investors and for the whole economy, are not well understood by financial scholars. Several years after October 1987, we are still wondering what caused the stock market breakdown. Moreover, it is difficult to match extreme price movements with rational

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explanations. Cutler, Poterba, and Summers (1989), analyzing large daily price movements from 1928 to 1987 to see whether they are related to the arrival of information, conclude that some extreme returns are not associated with major news stories.

The profusion of financial databases and the advent of the computer have made possible new approaches to the study of the stock market. Most of the empirical studies and models concern average properties like expected returns, volatility, or correlations, and little attention has been given to the extreme movements themselves. Notable exceptions are Mandelbrot (1963) and Fama (1965), who suggest the use of stable Paretian laws for returns to take into account the large number of outliers; Rothschild and Stiglitz (1970), who use the weight of the tails of two random variables to propose a better definition of increasing risk than the standard one (i.e., the usual variance); Parkinson (1980), who recognizes that extremes could contain information useful for a more efficient computation of the variance; McCulloch (1978), who studies the discontinuities of the price process associated with large falls and rises; Jansen and De Vries (1991), who use extremes to investigate the fatness of the distribution tails; and Loretan and Phillips (1994), who use the extremal index for testing the covariance stationarity of time series.

This article examines the extreme movements of the U.S. stock market over a century of daily observations (1885–1990). An extreme movement is defined as the lowest daily return (the minimum) and the highest daily return (the maximum) of the stock market index observed over a given period. Extremes are random variables depending on the distribution of returns and on the length of selection period. Extreme value theory shows that certain results which are distribution-free can be reached: the form of the asymptotic distribution of the extreme returns is independent of the process generating returns; only the distribution parameters' value depends on it. It is shown empirically in this study that the distribution of minima and maxima is a Fréchet distribution. It is accurately estimated and shown to fit well with data for an index of the most traded stocks on the New York Stock Exchange.

The remainder of the article is organized as follows: Section II presents extreme value theory; Section III reviews different methods of estimating extreme value distribution parameters; the empirical analysis starts in Section IV by estimating the asymptotic distribution of extreme returns, followed by results concerning the time stability of the distribution of extremes and the distribution of extremes obtained from lower frequency returns; finally, Section V sums up the study and outlines the economic implications and the applications of the results.

II. Theory of Extremes

This section presents exact and asymptotic statistical results pertaining to the theory of extremes. Recent advances in extreme value theory are also discussed. This presentation draws on Gumbel's (1958) book which gives an excellent exposition of the subject.[1]

A. *Exact Results*

Stock market price movements are measured by the daily logarithmic return of a stock market index denoted by X. Let us call f_X the probability density function, and F_X the cumulative distribution function of X. The support of the density function is denoted by (ℓ, u). Let X_1, $X_2, \ldots, X_n$ be the returns observed on days $1, 2, \ldots, n$. Extremes are defined as maxima and minima of the n random variables X_1, $X_2, \ldots, X_n$. Let Y_n denote the highest daily return (the maximum) observed over n trading days.[2] In the empirical study, from the first n observations of daily returns contained in the database $X_1, X_2, \ldots,$ X_n, one takes the largest observation denoted by $Y_{n,1}$. From the next n observations, $X_{n+1}, X_{n+2}, \ldots, X_{2n}$, another maximum called $Y_{n,2}$ is taken. From $n{\cdot}N$ observations of daily returns, one thus obtains N observed maxima, $Y_{n,1}, Y_{n,2}, \ldots, Y_{n,N}$. If the variables X are statistically independent and drawn from the same distribution (hypothesis of the random walk for stock market prices), then the exact distribution of the maximum Y_n can be written immediately as a function of the parent distribution F_X and the length of selection period n: $F_{Y_n}(x) = [F_X(x)]^n$. From this formula, it can be concluded that the limiting distribution of Y_n is null for x less than the upper bound u and equal to one for x greater than u. Such an exact expression is not, however, especially interesting since the exact limiting distribution is degenerate. In practice, the distribution of the parent variable is not precisely known and, therefore, if this distribution is not known, neither is the exact distribution of the extremes. For theoretical purposes as well as for practical ones, this study focuses on the asymptotic behavior of the extremes.

B. *A Limiting Result: The Extreme Value Theorem*

To find a limiting distribution of interest, the maximum variable Y_n is reduced with a location parameter β_n and a scale parameter α_n (as-

1. Galambos's (1978) textbook gives a detailed and rigorous account of the probabilistic aspects of extreme value theory. The textbook by Leadbetter, Lindgren, and Rootzèn (1983) gives advanced results for conditional processes.

2. The remainder of the article presents theoretical results for the maximum only, since the results for the minimum Z_n can be directly deduced from those of the maximum by transforming the random variable X into $-X$, by which maximum becomes minimum and vice versa. The following relation is used: $Z_n(X) \equiv \min(X_1, X_2, \ldots, X_n) = -\max(-X_1, -X_2, \ldots, -X_n) = -Y_n(-X)$.

sumed to be positive) such that the distribution of standardized extremes $(Y_n - \beta_n)/\alpha_n$ is nondegenerate. Gnedenko (1943) proves the so-called extreme value theorem which specifies the form of the limiting distribution F_Y as the length of the period over which extremes are selected tends to infinity. Three possible types of limiting extreme value distributions can be reached: the Gumbel distribution (type 1),

$$F_Y(y) = \exp(-e^{-y}) \quad \text{for } y \in \mathbb{R}, \tag{1}$$

the Fréchet distribution (type 2),

$$F_Y(y) = \begin{cases} 0 & \text{for } y \le 0 \\ \exp(-y^k) & \text{for } y > 0\,(k > 0), \end{cases} \tag{2}$$

and the Weibull distribution (type 3),

$$F_Y(y) = \begin{cases} \exp(-(-y)^{-k}) & \text{for } y < 0\,(k < 0) \\ 1 & \text{for } y \ge 0. \end{cases} \tag{3}$$

Gnedenko (1943) gives necessary and sufficient conditions for a particular distribution to belong to one of the three types: for type 1,

$$\lim_{n \to +\infty} n[1 - F_X(\alpha_n \cdot x + \beta_n)] = e^{-x}. \tag{C1}$$

For type 2,

$$\lim_{t \to +\infty} \frac{1 - F_X(t \cdot x)}{1 - F_X(t)} = x^k, \tag{C2}$$

where $t > 0$ and $k > 0$. Condition (C2) expresses the fact that the variable X varies regularly at infinity (Feller 1971, chap. 8). For type 3,

$$\lim_{t \to 0} \frac{1 - F_X(t \cdot x + u)}{1 - F_X(x + u)} = x^{-k}, \tag{C3}$$

where u is the end point of the distribution $X(F_X(u) = 1)$, $t > 0$, and $k < 0$.

The shape parameter k reflects the weight of the tail of the distribution of the parent variable X. The shape parameter k as well as the normalizing coefficients α_n and β_n may be different for minima and maxima. The tail of the distribution F_X is either declining exponentially (type 1) or by a power (type 2) or is finite (type 3). For the first and third cases, all moments of the distribution of X are well defined. For the second case the shape parameter k corresponds to the maximal order moment: the moments of order r greater than k are infinite, and the moments of order r less than k are finite (Gumbel 1958, p. 266); the distribution of X is fat-tailed. The lower k, the fatter the distribution of X; for example, if k is greater than unity, then the mean of the distribution exists; if k is greater than two, then the variance is finite, if k is greater than three, then the skewness is well defined, and so forth. The shape parameter is an intrinsic parameter of the process of daily returns and does not depend on the number of daily returns n from which the maximal return is selected.

Basic results from Gnedenko's (1943) article are that only distributions unbounded (to the right) can have a Fréchet distribution as limit, only distributions with finite right end point ($u < +\infty$) can have Weibull as limit, while the Gumbel distribution can be the limit of bounded or unbounded distributions.

Conditions (C1)–(C3) may be employed in specific cases to derive the type of asymptotic distribution of extremes. For example, the normal distribution commonly used in finance leads to the Gumbel distribution for the extremes. The Student-t distributions considered by Praetz (1972) obey the Fréchet distribution with a shape parameter k equal to its degree of freedom ($k \geq 2$). A stable Paretian law introduced by Mandelbrot (1963) also leads to a Fréchet distribution with a shape parameter k equal to its characteristic exponent ($0 < k < 2$).

Jenkinson (1955) proposes a generalized formula (4) which groups the three types distinguished by Gnedenko (1943):

$$F_Y(y) = \exp[-(1 - \tau \cdot y)^{1/\tau}] \begin{cases} \text{for } y > \tau^{-1} & \text{if } \tau < 0, \\ \text{for } y < \tau^{-1} & \text{if } \tau > 0. \end{cases} \tag{4}$$

The parameter τ called the tail index is related to the shape parameter k by $\tau = -1/k$. The tail index determines the type of distribution: $\tau < 0$ corresponds to a Fréchet distribution (type 2), $\tau > 0$ to a Weibull distribution (type 3), and the intermediate case ($\tau = 0$) corresponds to a Gumbel distribution (type 1). The Gumbel distribution can be regarded as a transitional limiting form between the Fréchet and the Weibull distributions, as $(1 - \tau \cdot y)^{1/\tau}$ is interpreted as e^{-y}. For small values of τ (or large values of k), the type 2 and type 3 distributions are very close to the type 1 distribution.

The extreme value theorem has been extended to time series; Berman (1964) shows that the same result stands if the variables are correlated (the sum of squared correlation coefficients remaining finite); Leadbetter, Lindgren, and Rootzèn (1983) consider various processes based on the normal distribution: autoregressive processes with normal disturbances, discrete mixtures of normal distributions as studied in Kon (1984), and mixed diffusion jump processes as advanced by Press (1967) all have thin tails so that they lead to a Gumbel distribution for the extremes; and De Haan et al. (1989) show that if X follows an ARCH process introduced by Engle (1982), then the maximum has a Fréchet distribution. For an ARCH(1) model defined by $X_t = \epsilon_t(e_0 +$

$e_1 \cdot X_{t-1}^2)^{1/2}$ where ϵ_t are independent and identically distributed as a $N(0,1)$, the shape parameter k is greater than two and obtained from the equation $\Gamma(k + 0.5) = \pi^{1/2}(2 \cdot e_1)^{-k}$.

III. Statistical Estimation Procedures

Estimated empirically, the asymptotic distribution of extremes contains three parameters only: τ, α_n, and β_n. A first approach, called parametric, consists of estimating these parameters by assuming that realized extremes are drawn *exactly* from this distribution. Two parametric techniques are commonly used: the maximum likelihood method which provides efficient estimates and the regression method which provides a graphical method for determining the type of asymptotic distribution. A second approach called nonparametric is based on the direct tail estimation of the parent variable X and does not assume that extremes are drawn exactly from the asymptotic distribution. These different methods should enable us to determine the asymptotic distribution of returns by many ways: statistical tests, graphical determination, and study of power.

A. *Parametric Approach*

1. *The maximum likelihood method.* The maximum likelihood method gives parameter estimators which are unbiased, asymptotically normal, and of minimum variance. Parameters' estimates are obtained by solving a set of nonlinear equations given by the first-order conditions of the maximization problem (see Tiago de Oliveira 1973). A likelihood ratio test will be computed to discriminate among the three types of asymptotic distributions of extremes.

2. *The regression method.* The regression method described in Gumbel (1958, pp. 226, 260, 296) is based on order statistics of the extremes Y. The sequence of observed maxima $(Y_{n,i})_{i=1,N}$ is arranged in increasing order to get an order statistic $(Y'_{n,i})_{i=1,N}$, for which: $Y'_{n,1} \leq Y'_{n,2} \leq \ldots \leq Y'_{n,N}$. For each value of i, the frequency $F_{Y,n}(Y'_{n,i})$ is a random variable lying between zero and one. The distribution of these variables is independent of the variable Y_n. The mean of the ith frequency $E[F_{Y,n}(Y'_{n,i})]$ is equal to $i/(N + 1)$. The method compares the ordered extreme observation $F_{Y,n}(Y'_{n,i})$ to its theoretical counterpart $i/(N + 1)$. This is done by estimating the reduced equation (5) obtained by twice taking the logarithm of both quantities:

$$-\ln\left[-\ln\left(\frac{i}{N+1}\right)\right] = \frac{1}{\tau}\cdot\ln\alpha_n - \frac{1}{\tau}\cdot\ln\left(-\tau\left(Y'_{n,i} - \beta_n - \frac{\alpha_n}{\tau}\right)\right) + \phi_{n,i}. \tag{5}$$

For the intermediate Gumbel case ($\tau = 0$), the following regression is run:

$$-\ln\left[-\ln\left(\frac{i}{N+1}\right)\right] = \frac{Y'_{n,i} - \beta_n}{\alpha_n} + \phi_{n,i}. \tag{6}$$

Consistent parameter estimates are obtained for both nonlinear equations (5) and (6) by minimizing the sum of squared residuals. A graphical test derived by Jenkinson (1955) allows the establishment of a preference for one of the three types of extreme value distribution (see also Gumbel 1958, p. 178). The theoretical values $-\ln[-\ln(i/(N+1))]$ are plotted against the observations of ordered extremes $Y'_{n,i}$ on probability paper. The curvature of the resulting graph is related to the type of distribution: for a Gumbel distribution, a straight line should be obtained. The Fréchet distribution leads to a concave curve, while the Weibull distribution gives a convex curve. Gumbel (1958, p. 215) gives the confidence bounds for the graphs.

B. Nonparametric Approach

Estimators for the tail index τ which do not assume that the observations of extremes follow exactly the asymptotic distribution have been developed by Pickands (1975) and Hill (1975). In this situation such estimators may be more efficient than maximum likelihood estimators as claimed by Jansen and De Vries (1991). These estimators are based on order statistics of the parent variable X. For the maximum they are given by formulae (7) and (8):

$$\tau_{Pickands} = -\frac{1}{\ln 2} \cdot \ln\left(\frac{X'_{N^{obs}-q+1} - X'_{N^{obs}-2q+1}}{X'_{N^{obs}-2q+1} - X'_{N^{obs}-4q+1}}\right), \tag{7}$$

and

$$\tau_{Hill} = \frac{1}{q-1} \cdot \sum_{i=1}^{q-1} (\ln X'_{N^{obs}-i} - \ln X'_{N^{obs}-q}), \tag{8}$$

where $(X'_m)_{m=1,N^{obs}}$ is the series of daily returns ranked in an increasing order and q (the number of tail observations to consider) is an integer depending on the number of observations of daily returns in the database N^{obs}. Pickands's estimator is consistent if q increases at a suitably rapid pace with N^{obs} (see Dekkers and De Haan 1989). Normalized Pickands's statistic $(\tau_{Pickands} - \tau) \cdot q^{1/2}$ is asymptotically normally distributed with mean zero and variance $\tau^2(2^{-2\tau+1} + 1)/[2(2^{-\tau} - 1)\ln 2]^2$. Pickands's estimator can be computed for all types of distribution and used for a t-test to discriminate among the three distributions of extremes and study the power of the test. Hill's estimator can be used in the case of the Fréchet distribution only ($\tau < 0$). In this situation, Hill's estimate is more efficient than Pickands's. Mason (1982) shows that Hill's estimate is consistent, and Goldie and Smith (1987) show that $(\tau_{Hill} - \tau)q^{1/2}$ is asymptotically normally distributed with mean zero and variance τ^2. Consistency is still obtained under weak dependence on the parent variable X. For both estimators an optimal value for q is computed by Monte Carlo simulations as suggested by Jansen and De Vries (1991).

IV. Empirical Analysis

A. Data

The database used is described in Schwert (1990*a*). The sample includes 29,641 daily observations of an index of the most traded stocks on the New York Stock Exchange. Using logarithmic daily percentage returns, the returns can take any value ranging from $-\infty$ to $+\infty$, and then any type of extreme value distribution can be obtained a priori.

The daily returns have a positive mean of 0.031% and a high standard deviation of 1.053% (in annual unit an average return of 8.70% and a volatility of 17.60%). The returns distribution is slightly skewed (-0.506) and presents excess kurtosis (22.057) which suggests departure from the normal distribution. The first-order autocorrelation is small (0.047) but significantly positive. Little serial correlation is found at higher lags. For the second moment, a strong positive serial correlation (0.229 at lag 1) is found, which suggests ARCH effects.

Some statistics for the extremes are as follows. Minima and maxima defined as the largest daily fall in the stock market and the largest daily rise over a year (containing on average 279 daily returns) are widely spread. For the largest declines, the minimal value (-22.90) is obtained in 1987 during the crash and the maximal value (-1.26) in 1964. For the largest rises, the minimal value (0.89) is obtained in 1964 and the maximal value (15.37) in 1933, after the bank holidays declared by President Franklin Roosevelt. A characteristic of the extremes is their clustering: there are 28 years (from among 106) during which the minima and the maxima occur in the same week. In general, the price decrease precedes the price increase.

B. Asymptotic Behavior of the Distribution of the Extreme Returns

The asymptotic distributions of minima and maxima selected over nonoverlapping periods of varying length, from 1 month to 2 years, are estimated first. This enables examination of the way the asymptotic distribution coverges. As the number of days n from which extremes are selected increases (from n = 23 to n = 559), one should observe (1) the stability of the tail index around a particular value, since it is an intrinsic parameter of the process of returns; (2) an increase in absolute value of the location parameter, since extremes selected over longer periods are automatically larger; and (3) the behavior of the scale parameter is not specified a priori as the distribution of extremes may contract or expand (Gumbel 1958, p. 154).

A graphical representation of the asymptotic behavior of parameters' estimates is given in figure 1*a* for the minimum and figure 1*b* for the maximum, while panels A and B of table 1 give the estimates' values obtained with the maximum likelihood method for the following lengths of the selection period: 1 month, 1 quarter, 1 semester, 1 year, and 2 years.

The tail index is stable, especially for selection periods longer than a semester, for both types of extremes, as the value seems to converge around -0.40 for minima and -0.35 for maxima. Nonparametric estimates of the tail index are also computed (table 2) using all the relevant information contained in the tails. Pickands's estimates are very close to the ones obtained by the parametric method: -0.415 for minima and -0.380 for maxima.

For both types of extremes, the location parameter increases in absolute value as expected. For example, the location parameter for the minimum observed over 1 month is -1.193%, while it equals -3.185% for the longer time period of 2 years.

As the tail index is negative and greater than minus one, the scale parameter is expected to increase with the length of selection period (Gumbel 1958, p. 154). It is indeed the case for both extremes. For example, for the minima, it increases from 0.623 to 1.569.

To complete the statistical analysis of the asymptotic behavior of the distribution of extremes, the quality of convergence of the extreme value distribution is assessed by carrying out a numerical test. It is desirable to estimate the error resulting from the replacement of the exact distribution of extremes by the limiting one, in other words the speed of convergence of the asymptotic distribution of extremes. The Sherman (1957) test for goodness of fit is used here to compare the probability given by the asymptotic distribution to the observed frequency used as a proxy of the exact probability.[3] Such an exercise combines two kinds of errors: the sampling error due to the limited number of observations and the error due to the incompleteness of the passage to the limit as the length of selection period n increases indefinitely. The test statistic is computed as follows: $\Omega_N = \frac{1}{2}\Sigma_{i=0}^{N}|F_{Y,n}(Y'_{n,i}) - F_{Y,n}(Y'_{n,i+1}) - 1/(N+1)|$, with $F_{Y,n}(Y'_{n,0}) = 0$ and $F_{Y,n}(Y'_{n,N+1}) = 1$. The variable Ω_N is asymptotically normal, with

3. Sherman's (1957) test has good small-sample properties as it quickly converges toward normality. It is also more suitable than the Kolmogorov-Smirnov and chi-squared tests since it does not require the arbitrary division of data into groups (Gumbel 1958, p. 38).

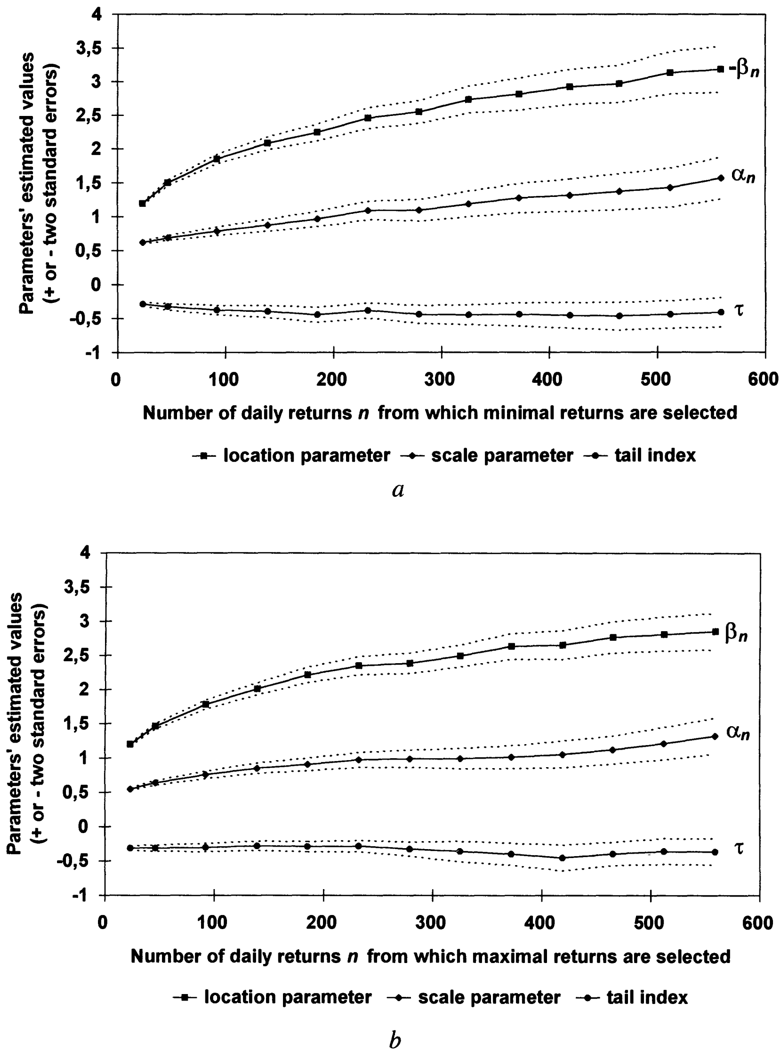


FIG. 1.—Asymptotic behavior of the parameters of the distribution of extreme returns. *a*, minimal returns. *b*, maximal returns. Extreme returns are selected over nonoverlapping periods of varying length: from 1 month (n = 23) to 2 years (n = 559). Estimates of the location parameter, the scale parameter, and the tail index are obtained by the maximum likelihood method.

mean $(N/(N + 1))^{N+1}$ and variance $(2e - 5)/(e^2N)$. The result of the test is reported in the last column of table 1. As the length of selection period n increases, the asymptotic distribution describes the behavior of extremes better: the goodness of fit cannot be rejected at the one-per-thousand confidence level for extreme returns selected over a semester, nor at the 10% level for extremes selected over longer periods.

TABLE 1 Asymptotic Behavior of the Distribution of Extreme Returns

Length of Selection Period	Scale Parameter α_n	Location Parameter β_n	Tail Index τ	Sherman's Goodness-of-Fit Statistics (p-Value)
A. Minimal returns:				
1 month (n = 23, N = 1,288)	.623	−1.193	−.285	53.587
	(.011)	(.013)	(.016)	($p < 10^{-5}$)
1 quarter (n = 69, N = 429)	.771	−1.701	−.322	8.964
	(.025)	(.029)	(.026)	($p < 10^{-5}$)
1 semester (n = 139, N = 213)	.873	−2.084	−.394	3.264
	(.043)	(.047)	(.043)	(p = .001)
1 year (n = 279, N = 106)	1.092	−2.549	−.441	0.411
	(.079)	(.085)	(.067)	(p = .681)
2 years (n = 559, N = 53)	1.569	−3.185	−.413	−1.257
	(.155)	(.172)	(.109)	(p = .208)
B. Maximal returns:				
1 month (n = 23, N = 1,288)	.552	1.201	−.309	58.444
	(.010)	(.012)	(.016)	($p < 10^{-5}$)
1 quarter (n = 69, N = 429)	.706	1.630	−.306	13.370
	(.022)	(.027)	(.027)	($p < 10^{-5}$)
1 semester (n = 139, N = 213)	.856	2.009	−.277	2.506
	(.037)	(.045)	(.034)	(p = .012)
1 year (n = 279, N = 106)	.993	2.385	−.323	.628
	(.064)	(.075)	(.051)	(p = .530)
2 years (n = 559, N = 53)	1.329	2.857	−.359	−.977
	(.131)	(.133)	(.096)	(p = .328)

NOTE.—This table gives the parameters' maximum likelihood estimates of the distribution of extreme returns. Asymptotic standard errors are given in parentheses. A minimal return (panel A) corresponds to the lowest daily return reached over time periods of different length: 1 month, 1 quarter, 1 semester, and 1 and 2 years. A maximal return (panel B) corresponds to the highest daily return over these periods. The number of daily returns n from which extreme returns are selected and the resulting number of extreme observations N are given in the first column. The last column reports Sherman's (1957) goodness-of-fit statistic for the distribution of extremes, with the p-value in parentheses.

TABLE 2 Nonparametric Estimates of the Tail Index

Variable	Pickands's Estimate	Hill's Estimate
Minimal returns	−.415	−.361
	(.093)	(.027)
Maximal returns	−.380	−.330
	(.094)	(.024)

NOTE.—This table gives nonparametric estimates of the tail index obtained by Hill's (1975) and Pickands's (1975) formulae. Asymptotic standard errors are given in parentheses. The optimal number of tail observations q used to compute these estimates is found by simulation. It is equal to 416 for Pickands's formula and to 179 for Hill's formula for both minimal and maximal returns.

For yearly extreme returns, the p-values of Sherman's test are rather high: 0.681 for minima and 0.530 for maxima. The distribution of extremes converges relatively quickly as the length of selection period n increases, and the asymptotic formula can be used reliably for extremes selected over periods longer than a semester.

C. The Type of Asymptotic Distribution of the Extreme Returns

Turning now to the determination of the type of extreme value distribution: as the statistical behavior of extremes selected over periods longer than a semester seems well represented by the asymptotic distribution, let us concentrate on yearly minima and maxima. Table 3 gives maximum likelihood estimates of the constrained Gumbel case and of the unconstrained case for the asymptotic distribution of extremes. The empirical results are clear-cut and allow one to determine umambiguously the type of asymptotic distribution: for both the yearly largest falls and rises, the asymptotic distribution belongs to the domain of attraction of the Fréchet distribution. Both cases give a tail index τ significantly different from zero. For the minima, the estimate of τ is equal to -0.441, with a t-ratio equal to -6.68. For the distribution of the maxima, the estimate of τ is equal to -0.323, with a t-ratio equal to -6.33. Equivalently, the shape parameter values are 2.266 (0.344) for minima and 3.094 (0.489) for maxima. A likelihood ratio test be-

TABLE 3 **Maximum Likelihood Estimates of the Scale Parameter, Location Parameter, and Tail Index of the Asymptotic Distribution of Yearly Extreme Returns**

Variable	Scale Parameter α_n	Location Parameter β_n	Tail Index τ	Log Likelihood Value (LR Test, p-Value)
Minimal returns, Gumbel distribution	1.496	-2.854	0	-221.679
	(.153)	(.113)	...	...
Minimal returns, Fréchet distribution	1.092	-2.549	$-.441$	-203.448
	(.079)	(.085)	(.067)	(LR = 36.46; $p < 10^{-5}$)
Maximal returns, Gumbel distribution	1.238	2.580	0	-201.922
	(.126)	(.093)	...	...
Maximal returns, Fréchet distribution	.993	2.385	$-.323$	-186.872
	(.064)	(.075)	(.051)	(LR = 30.10; $p < 10^{-5}$)

NOTE.—This table gives parameters' parametric estimates of the distributions of extreme returns. Asymptotic standard errors are given in parentheses. Minimal (maximal) return corresponds to the lowest (highest) daily return reached over a year containing on average n (= 279) trading days over the period 1885–1990. Estimates of the three parameters (α_n, β_n, and τ) are obtained using the maximum likelihood method and reported for the constrained Gumbel distribution ($\tau = 0$) and for the unconstrained Fréchet distribution. The statistic of the likelihood ratio test (LR) between the two models is reported in the last column, with the p-value in parentheses. The test is asymptotically distributed as a chi-square with 1 degree of freedom.

tween the Fréchet case and the Gumbel case leads to a firm rejection of the Gumbel distribution (and a fortiori a rejection of the Weibull distribution). The test value is equal to 36.46 for the minima and 30.10 for the maxima, with p-values less than 10^{-5}.

These estimations, tests, and conclusions are similar to those given by the regression method. The value of the objective function of nonlinear equation (5) is greatly improved when the constraint ($\tau = 0$) is relaxed in equation (6). For minima the sum of squared residuals of the Gumbel equation (6) is equal to 27.680, while it is equal to 1.756 for the Fréchet equation (5) with an unconstrained tail index of -0.455. For the maxima, the corresponding sums of squared residuals equal 22.369 and 1.723, with an unconstrained tail index of -0.414. Additionally, the graphical test described in Subsection III*A* is performed. Figures 2*a* and 2*b* plot the ordered extreme returns against the theoretical values. The values predicted by the Gumbel distribution lie on the straight lines in figure 2*a* for minima and figure 2*b* for maxima. Clearly, the data do not lie close to these straight lines, as they would if the extremes were drawn from a Gumbel distribution; around 30% of the observations lie outside the confidence bounds. The curvature is rather concave, which suggests that the limiting distribution is a Fréchet distribution. Figures 3*a* and 3*b* give a graphical representation of the fit of the Fréchet distribution with the data. The observations lie close to a straight line, which confirms the good fit of the Fréchet distribution with the data; all observations but one lie inside the 1 standard deviation confidence bounds. For the minima, even the great crashes of October 1929 and 1987 are close to their predicted values. This tends to dismiss the view of the crashes as singular events.[4]

Nonparametric estimates of the tail index reported in table 2 also lead to the rejection of the Gumbel in favor of the Fréchet distribution. Pickands's estimator (valid for all types of distribution) produces negative values which are significantly different from zero ($t = -4.46$ for minima and $t = -4.08$ for maxima). The tests based on Pickands's estimator are quite powerful as shown now. For the null hypothesis defined by H_0: $\tau = 0$ and a type 1 risk fixed at 5%, the type 2 risk (i.e., the acceptance of the Gumbel case while it is in fact not true) can be computed. This requires the definition of an alternative hypothesis. The most general alternative hypothesis is defined by the compos-

4. From a statistical point of view, this empirical result is not surprising. As suggested by the referee, most stock market crashes can be considered as extremes of extremes. According to the theory, the distribution of extremes of extremes is of the same type as the distribution of extremes itself (see Gumbel [1958], pp. 157–62, for an exposition of the so-called stability postulate). From condition (C2) it can be directly verified that extremes drawn from a Fréchet distribution also follow a Fréchet distribution with the same shape parameter value.

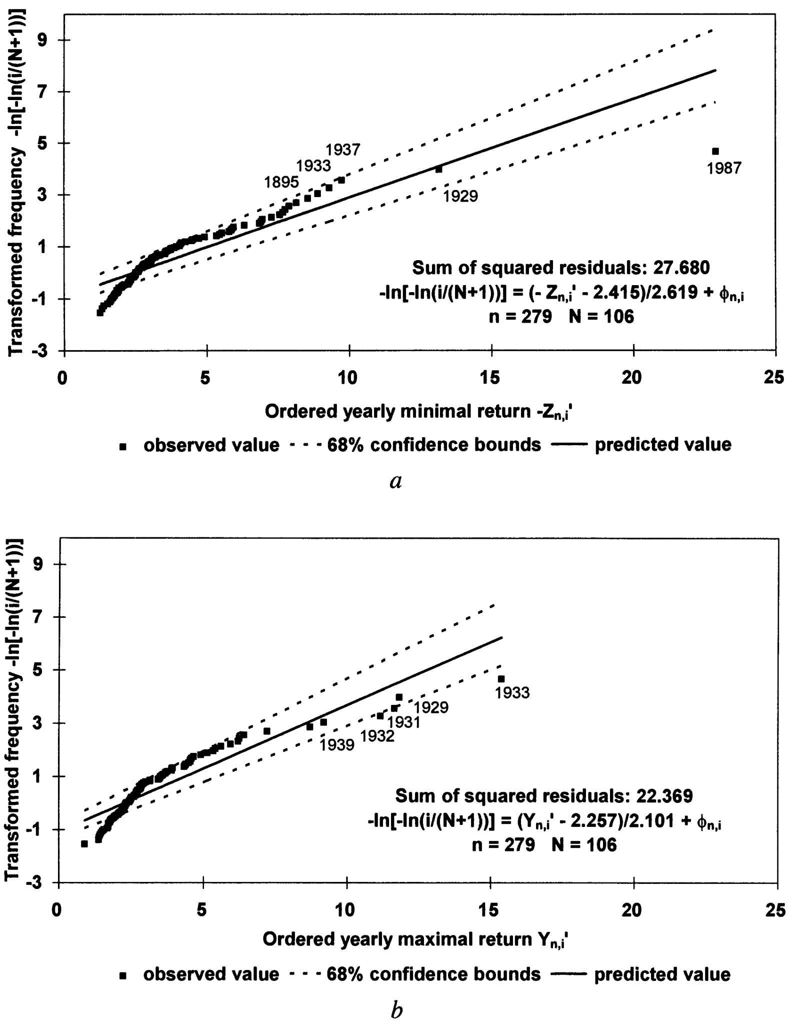


FIG. 2.—Graphical determination of the type of asymptotic distribution for yearly extreme returns. *a*, minimal returns. *b*, maximal returns. If extreme returns were drawn from a Gumbel distribution, observed ordered extremes should lie on a straight line. Concavity suggests a Fréchet distribution.

ite hypothesis H_1: $\tau \neq 0$, but it is necessary to consider simple alternative hypotheses in order to get a specific value for the error of the second type. Table 4 gives the type 2 risks and the power of the test for various alternative hypotheses corresponding to fat-tailed distributions with different tail index values. The lower the tail index value, the higher the power of the test as the Fréchet distribution moves away from the Gumbel distribution. For a tail index value similar to those found empirically, the power is quite high at around 80%.

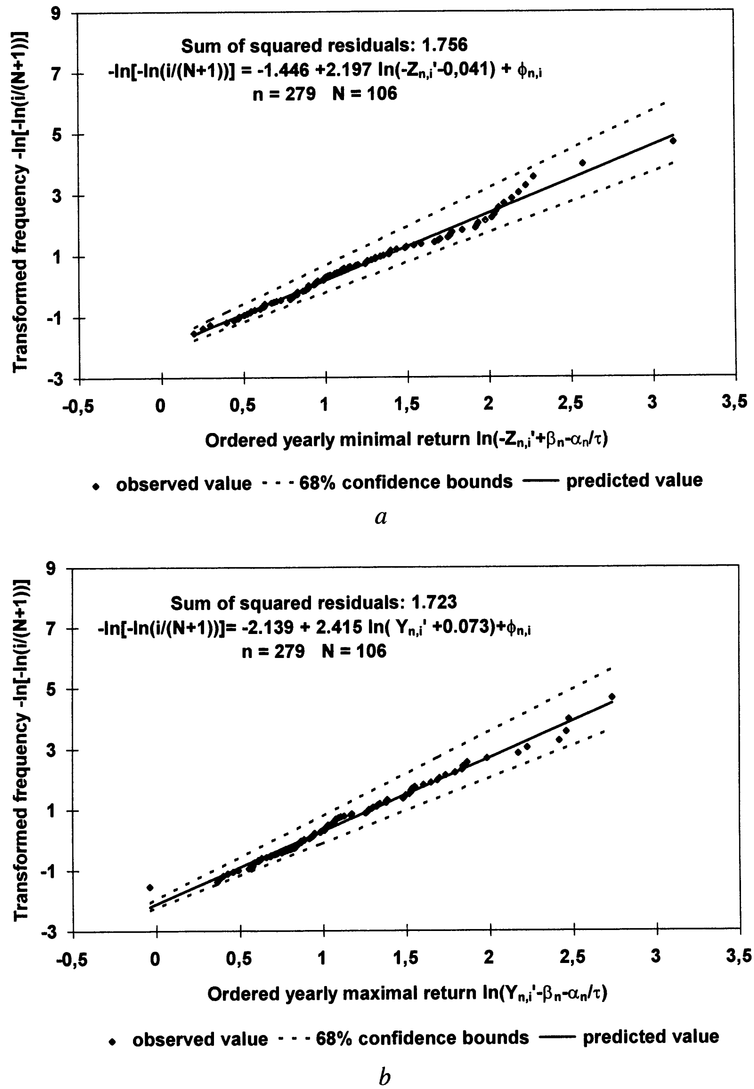


FIG. 3.—Adjustment of the Fréchet distribution to yearly extreme returns. *a,* minimal returns. *b,* maximal returns.

Hill's estimator can also be used as the tail index is very likely negative. Similar values are obtained: -0.361 for minima and -0.330 for maxima. Standard errors are lower since Hill's estimator is more efficient than Pickands's estimator in the case of the Fréchet distribution.

In sum, as extremes are selected over a longer time period, the distribution of extremes is a Fréchet distribution which shifts to the

TABLE 4 Power of the Test ($\tau = 0$) Based on Pickands's Estimator

	For Alternative Hypothesis $H_{1,\tau}$, Shape Parameter k, and Optimal Value for q				
	$\tau = -.10$ $k = 10$ $q = 239$	$\tau = -.25$ $k = 4$ $q = 298$	$\tau = -.33$ $k = 3$ $q = 416$	$\tau = -.50$ $k = 2$ $q = 535$	$\tau = -1.00$ $k = 1$ $q = 1,128$
Type 2 risk (%)	92.22	55.47	22.14	.08	.00
Power of the test (%)	7.78	44.53	77.86	99.92	100.00

NOTE.—This table gives the type 2 risk and the power of the test of the tail index value τ for various alternative hypotheses $H_{1,\tau}$. The null hypothesis is defined by H_0: $\tau = 0$ and corresponds to the Gumbel case. The type 1 error of 5% corresponds to a critical level of -0.264. For each hypothesis the optimal value of q used to compute the variance of Pickands's estimator $\tau^2(2^{-2\tau+1} + 1)/[2(2^{-\tau} - 1)\ln 2]^2/q$ is obtained by simulation. For the case ($\tau = 0$) 179 tail observations are used.

right for maxima and to the left for minima and expands, while the shape of the distribution remains the same. This behavior is graphically represented in figure 4. As suggested by this figure, the distributions of minima and maxima appear quite symmetric. This is confirmed by a statistical test: whether the distributions of minima and maxima are symmetric can be tested by directly comparing the coefficients from the two distributions: $\alpha_n^{\min} = \alpha_n^{\max}$, $\beta_n^{\min} = \beta_n^{\max}$, and $\tau^{\min} = \tau^{\max}$. Using yearly extremes, the values of the tests of equality are equal to 0.964

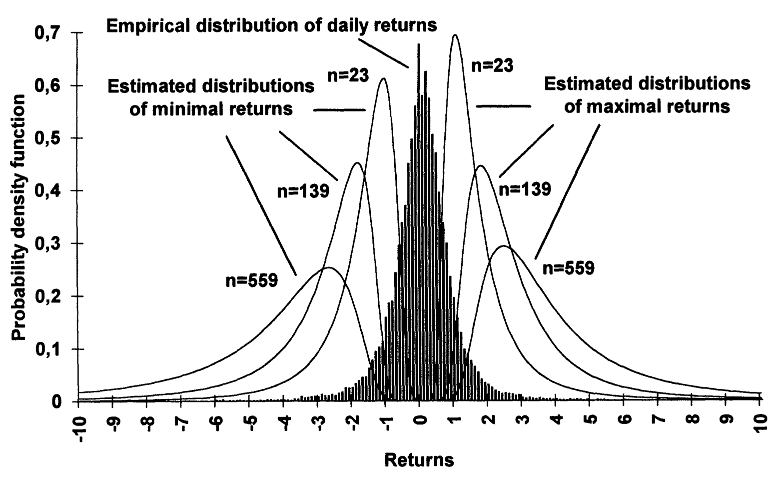


FIG. 4.—Asymptotic behavior of the distributions of extreme returns. Minimal and maximal returns are selected over periods of different length: 1 month ($n = 23$), 1 semester ($n = 139$), and 2 years ($n = 559$). Maximum likelihood estimates given in table 3 are used to compute the asymptotic distributions of extreme returns. As extremes are selected over a longer time period, the Fréchet distribution of extremes shifts to the left for minima and to the right for maxima and expands, while the shape of the distribution remains the same.

($p = 0.334$) for the scale parameter, 1.438 ($p = 0.150$) for the location parameter, and -1.404 ($p = 0.160$) for the tail index. For the three parameters, the equality cannot be rejected at standard confidence levels. This result is robust to the choice of estimator and remains valid across subperiods.

Tail index estimates can be used to determine the maximal order moment (highest finite moment) of the distribution of returns. As noted earlier, if the distribution of returns belongs to the domain of attraction of a Weibull or Gumbel extreme value distribution (let us say for the maximum), then *all* moments (truncated to the left) exist: $\int_0^{+\infty} x^r dF_X(x)$ is finite for all r. If the distribution of returns belongs to the domain of attraction of a Fréchet extreme value distribution with a tail index $\tau\,(= -1/k)$, then only *some* truncated moments exist; more precisely: $\int_0^{+\infty} x^r dF_X(x)$ is finite for $r \leq k$ and infinite for $r > k$. We have already seen that the null hypothesis H_0: $\tau \geq 0$ is strongly rejected, indicating that not all moments are defined and that the distribution of returns is fat-tailed. One now tests the null hypotheses H_0: $\tau \geq \tau^*$, where τ^* takes the values -0.25, -0.33, -0.50, and -1 to determine how fat-tailed the distribution is. With these selected values, one tests whether the kurtosis, the skewness, the variance, and the mean of the distribution of returns are defined. Results reported in table 5 for the left and right tails, obtained with three different estimators, homogeneously lead to the following conclusions: the mean is certainly well defined, as τ is greater than -1 with a probability very close to one;

TABLE 5 **Determination of the Maximal Order Moment of the Distribution of Returns Based on Tail Index Estimates**

	Null Hypothesis H_0: $\tau \geq \tau^*$ (Equivalently, $k \geq k^*$)				
Extreme and Estimator	$\tau \geq 0$ $k = +\infty$	$\tau \geq -.25$ $k \geq 4$	$\tau \geq -.33$ $k \geq 3$	$\tau \geq -.50$ $k \geq 2$	$\tau \geq -1.00$ $k \geq 1$
Left-tail Pickands	−4.462	−1.774	−.913	.914	6.290
	(.000)	(.038)	(.180)	(.819)	(1.000)
Right-tail Pickands	−4.082	−1.383	−.531	1.276	6.595
	(.000)	(.083)	(.297)	(.899)	(1.000)
Left-tail Hill	−13.370	−4.111	−1.148	5.148	23.666
	(.000)	(.000)	(.125)	(1.000)	(1.000)
Right-tail Hill	−13.750	−3.333	.000	7.083	27.916
	(.000)	(.000)	(.500)	(1.000)	(1.000)
Left-tail ML	−6.681	−2.893	−1.681	.894	8.469
	(.000)	(.002)	(.046)	(.814)	(1.000)
Right-tail ML	−6.333	−1.431	.137	3.471	13.274
	(.000)	(.076)	(.554)	(1.000)	(1.000)

NOTE.—This table gives the t-test for the null hypotheses H_0: $\tau \geq \tau^*$ where τ^*, takes the values 0, −.25, −.33, −.5, and −1. These tail index values correspond to shape parameter values k^* of $+\infty$, 4, 3, 2, and 1, respectively. Estimates and standard errors used for the test come from table 2 for Pickands's and Hill's estimators and from table 3 for the maximum likelihood (ML) estimator. The p-value in parentheses indicates the probability that the tail index τ is greater than τ^* or, equivalently, the probability that the moment of order k^* is finite.

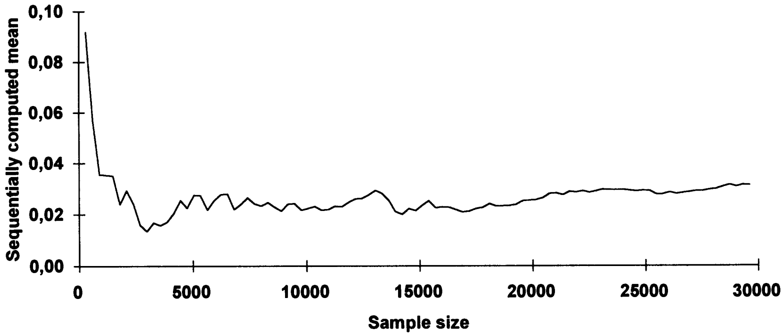
the variance is also likely defined, as τ is in most of the cases greater than the critical -0.50 with a high probability (it ranges from 81% to almost 100%); as for the skewness, the results are less clear as the probability of τ being greater than -0.33 is small but not negligible (it ranges from 4.6% to 55%); and the kurtosis is likely not defined, as the null hypothesis H_0: $\tau \geq -0.25$ is always rejected at the 10% level and sometimes at the 1% level.

These conclusions based on extreme values are in line with "converging moment" tests suggested by Mandelbrot (1963). The four first moments are computed sequentially by using more and more observations. If theoretical moments are finite, then corresponding empirical moments should converge. The sequential mean, variance, skewness, and kurtosis are plotted in figure 5. While the first and second moments seem to stabilize, the third and fourth moments tend to diverge as the sample size is increased. The sequential mean and variance show less erratic behavior than do the sequential skewness and kurtosis. For example, the crash of 1987 occurring at the end of the period has a small effect on the mean and the variance but produces a big jump in the skewness and kurtosis, although many observations have already been used to compute these moments.

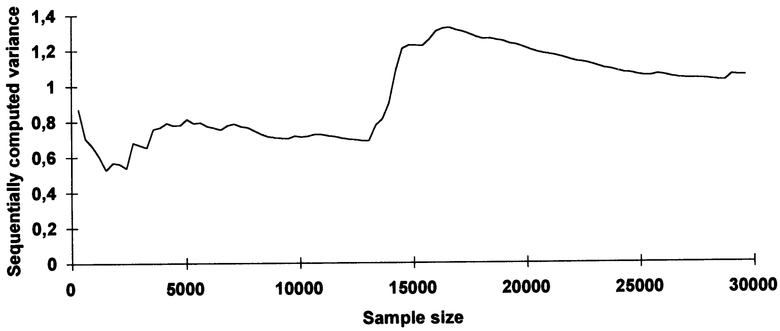
D. *Time Stability of the Asymptotic Distribution of Extremes*

The behavior of the asymptotic distribution over time can be examined by dividing the whole period 1885–1990 into 5 subperiods as in Schwert (1990*a*) and by estimating for each subperiod the parameters of the asymptotic distribution. Empirical results are reported in panel A of table 6 for minima and panel B of table 6 for maxima. Clearly, the parameters cannot be as accurately estimated as those for the whole period, since for each subperiod the number of observations is far fewer. A test is conducted for each coefficient separately: $\alpha_n^1 = \alpha_n^2$, $\beta_n^1 = \beta_n^2$, and $\tau^1 = \tau^2$, where the indexes 1 and 2 refer to subperiods. The null hypothesis of equality for adjacent subperiods is rejected only once at the 1% level: the period 1928–47 containing the stock market boom of the late twenties and the Great Depression seems different from the others. For both minima and maxima, the location and scale parameters are much higher, indicating that extreme returns were much larger and more widely dispersed during this period. The tail

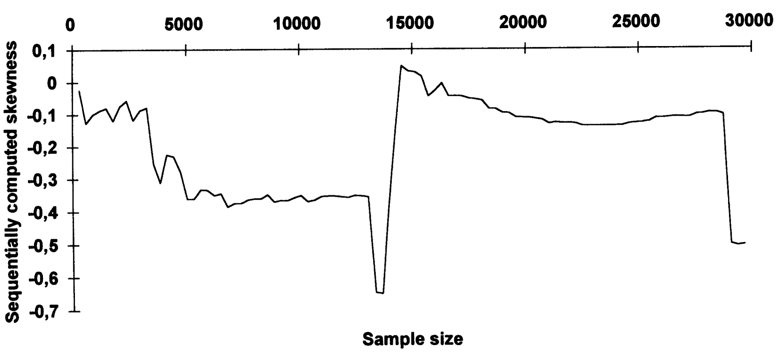
FIG. 5.—Converging moment tests. *a*, Behavior of the mean of the distribution of daily returns. *b*, Behavior of the variance of the distribution of daily returns. *c*, Behavior of the skewness of the distribution of daily returns. *d*, Behavior of the kurtosis of the distribution of daily returns. The first four moments (mean, variance, skewness, and kurtosis) are sequentially computed. Stability of the empirical moment suggests that the corresponding theoretical moment is finite, while an erratic behavior suggests that it is infinite.


Sequentially computed mean
0,10
0,08
0,06
0,04
0,02
0,00
0
5000
10000
15000
20000
25000
30000
Sample size

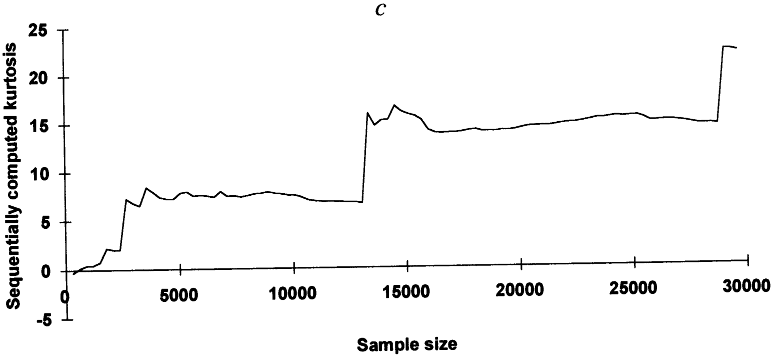
a


Sequentially computed variance
1,4
1,2
1
0,8
0,6
0,4
0,2
0
0
5000
10000
15000
20000
25000
30000
Sample size

b


0
5000
10000
15000
20000
25000
30000
Sequentially computed skewness
0,1
0
-0,1
-0,2
-0,3
-0,4
-0,5
-0,6
-0,7
Sample size

c


Sequentially computed kurtosis
25
20
15
10
5
0
-5
0
5000
10000
15000
20000
25000
30000
Sample size

d

index, however, is much more stable than the two other coefficients, suggesting that the shape of the tails does not change over time. This result confirms and extends over a much longer period the finding of Loretan and Phillips (1994, p. 242) who found the shape parameter to be stable over the recent period 1962–87.

The results for the extremes support the hypothesis of the stability of the statistical process for daily returns. Schwert (1990*a*) comes to the same conclusion when he studies other features of the database: the mean and the standard deviation of the daily returns and the seasonal pattern. As with Schwert, one can only be surprised by "the remarkable homogeneity for the series through time." This is surprising because many changes have occurred since the end of the last century: changes in the U.S. economy, in the trading environment, in the technology which transmits information, in the regulation of the securities markets, and so forth.

E. *The Asymptotic Distribution of Extreme Returns under Temporal Aggregation*

Finally, the distribution of extremes selected from basic returns of different frequency is investigated by considering time-aggregated returns corresponding to investment periods of 1 day, 1 week, and 1 month. Feller (1971, p. 279) shows that if $1 - F_X(x)$ varies regularly at infinity (i.e., verifies condition [C2]), then the maximum of any convolution follows the same limit law. This proposition specifies our understanding of the asymptotic behavior of time-aggregated returns. From the central limit theorem one already knows that values around the center of the distribution of time-aggregated returns are drawn asymptotically from a normal law if the variance is finite or from a stable Paretian law if the variance is infinite. Extreme value theory specifies the behavior of the tails of the time-aggregated distribution as it shows that they are stable under aggregation. Applied to finance, Feller's interesting mathematical result says that standardized extremes from returns with different frequencies are drawn from Fréchet distributions with the same tail index value. However, the scale and location parameters of the distribution of observed extremes can vary.

Empirical results are presented in panel A of table 7 for minima and panel B of table 7 for maxima. Basic returns are computed with a frequency of 1 day, 1 week, and 1 month. Extremes are selected in two different ways: (1) such that the number of extreme observations is kept constant ($N = 56$) which allows the direct comparison of parameters' estimates; and (2) such that the number of basic returns from which extremes are selected is kept constant ($n = 23$). The first method holds constant the sampling error, while the second holds constant the error due to the passage to the limit. Looking at the tail index, which should remain invariant under temporal aggregation, both methods of

TABLE 6 Behavior of the Distribution of Extreme Returns over Time

Subperiod	Maximum Likelihood Estimates: Scale Parameter α_n	Location Parameter β_n	Tail Index τ	Hill's Estimate, Tail Index τ
A. Minimal returns:				
1885–1906	.739	−1.980	−.376	−.285
	(.076)	(.086)	(.092)	(.033)
1907–27	.714	−2.018	−.191	−.295
	(.066)	(.085)	(.080)	(.041)
1928–47	1.519**	−3.622**	−.233	.315
	(.157)	(.192)	(.104)	(.032)
1948–67	.582**	−1.775**	−.344	−.315
	(.065)	(.075)	(.095)	(.032)
1967–90	.679	−1.845	−.517	−.293
	(.083)	(.086)	(.113)	(.034)
B. Maximal returns:				
1885–1906	.646	2.012	−.230	−.299
	(.059)	(.073)	(.075)	(.036)
1907–27	.592	1.818	−.200	−.244
	(.056)	(.071)	(.085)	(.034)
1928–47	1.345**	2.858**	−.542	.332
	(.170)	(.172)	(.124)	(.036)
1948–67	.585**	1.586**	−.093*	−.273
	(.056)	(.075)	(.080)	(.041)
1967–90	.831	2.085	−.115	−.239
	(.077)	(.102)	(.076)	(.034)

NOTE.—This table gives the parameters' maximum likelihood estimates for the distribution of extreme returns obtained over 5 subperiods. Asymptotic standard errors are given in parentheses. The number of basic returns n from which extreme returns are selected is constant, at 179. Nonparametric Hill's estimate of the tail index is reported in the last column. Optimal values found by simulation are equal to 67, 51, 49, 95, and 71 for minima and to 67, 51, 84, 43, and 59 for maxima.

* Significantly different from the parameter of the previous subperiod at the 5% level.

** Significantly different from the parameter of the previous subperiod at the 1% level.

selection give a similar result: the tail index is stable across frequency. For minima, tail index values obtained with the maximum likelihood method are very close to, and not statistically different from, each other: −0.285, −0.293, and −0.355. Similar comments apply to maxima: −0.309, −0.254, and −0.287. Hill's nonparametric estimates confirm these results. The scale and location parameters increase with the length of the investment period, suggesting that extremes from time-aggregated returns become larger and more dispersed. This was expected since the distributions of time-aggregated returns themselves become more dispersed by application of the central limit theorem.

V. Conclusion

This study concerns the extreme price movements of the U.S. stock market. The statistical distribution of minimal and maximal returns,

TABLE 7 Behavior of the Distribution of Extreme Returns under Temporal Aggregation

Return Frequency of Basic Returns	Maximum Likelihood Estimates: Scale Parameter α_n	Location Parameter β_n	Tail Index τ	Hill's Estimate, Tail Index τ
A. Minimal returns:				
1 day (n = 23, N = 1,288)	.623	−1.193	−.285	
	(.011)	(.013)	(.016)	−.361
1 day (n = 529, N = 56)	1.461	−3.184	−.426	(.027)
	(.148)	(.162)	(.103)	
1 week (n = 23, N = 257)	1.396	−2.963	−.293	
	(.059)	(.070)	(.040)	−.336
1 week (n = 105, N = 56)	2.080	−5.223	−.244	(.043)
	(.180)	(.221)	(.077)	
1 month (n = 23, N = 56)	3.201	−6.721	−.355	−.338
	(.307)	(.352)	(.095)	(.050)
B. Maximal returns:				
1 day (n = 23, N = 1,288)	.552	1.201	−.309	
	(.010)	(.012)	(.016)	−.330
1 day (n = 529, N = 56)	1.127	2.854	−.442	(.024)
	(.115)	(.125)	(.102)	
1 week (n = 23, N = 257)	1.060	2.881	−.254	
	(.042)	(.052)	(.032)	−.255
1 week (n = 105, N = 56)	1.564	4.310	−.286	(.027)
	(.154)	(.181)	(.114)	
1 month (n = 23, N = 56)	2.386	7.113	−.287	−.270
	(.206)	(.249)	(.067)	(.057)

NOTE.—This table gives the parameters' maximum likelihood estimates for the distribution of extreme returns obtained from returns of frequency equal to 1 day, 1 week, or 1 month. Asymptotic standard errors are given in parentheses. The number of basic returns n from which extreme returns are selected and the resulting number of extreme observations N are given in the first column. Results are reported for two methods of selection of extremes: (1) n is kept constant and equal to 23, and (2) N is kept constant and equal to 56. Nonparametric Hill's estimate is also reported in the last column. Optimal values found by simulation are equal to 179, 61, and 45 for minima and to 179, 84, and 22 for maxima.

defined as the lowest and highest daily return over a given period, is estimated using extreme value theory. Statistical theory states that the asymptotic distribution of extremes has a well-determined form which is independent of the process of returns. The major findings are:

1. The asymptotic distribution of extreme returns is a Fréchet distribution. As extremes are selected over a longer time period, the distribution of extremes shifts to the right for maxima and to the left for minima and expands, while the shape of the distribution remains the same.

2. The tail index value allows one to determine the degree of fatness of the distribution of returns: the mean and the variance are certainly well defined, while the skewness, the kurtosis, and all higher order moments may be infinite.

3. The results are fairly stable over time, although the period of the Great Depression exhibits larger and more dispersed extremes. The

shape of the tails, however, seems constant over the entire period, even during the thirties.

4. The distribution of extremes is also found to be stable under temporal aggregation. Extremes selected from daily, weekly, and monthly returns follow a Fréchet distribution.

With regard to the economic implications and applications of these results, Fama (1963) discusses two extreme cases: the discontinuous stable Paretian hypothesis and the continuous Gaussian hypothesis. In a stable Paretian market, a large price change over a long interval is, most of the time, the result of one or a few very large price changes that took place during smaller subintervals, and the price path contains discontinuities. In a Gaussian market, a large price change is more likely the result of many very small price changes, and the price path is continuous. This study of the U.S. market over a long period rejects both hypotheses (the tail index is significantly different from 0 and -0.5) and suggests an intermediate situation (the tail index is between 0 and -0.5). The market under study—a Fréchet market—presents more extremes and so more risk for investors than a Gaussian market but fewer extremes and so less risk than a stable Paretian market. The market price may or may not exhibit discontinuities according to the process governing returns.[5] Such a market characteristic has a direct economic implication for investors following stop-loss, arbitrage, or portfolio insurance strategies: in the case of continuity, these strategies may be as reliable as in a Gaussian market, although in practice larger price movements may occur on a short interval, and in the case of discontinuity, these strategies may be more efficient than in a stable Paretian market as large price movements occur less often. In a Fréchet market investors may have to use specific instruments to protect their position during high volatility periods. Longin (1996) has suggested the use of boom options and crash options to insure investors' portfolios against extreme price movements.

The results shed new light on the statistical process of returns. The behavior of extreme returns could be used to improve our understanding about the whole process. The tail index can be used to choose a model of returns from among those encountered in the financial literature: normal processes, Student-*t* distributions, ARCH processes, and stable Paretian laws. As the weight of the tails is different for the above distributions, different values for the tail index are obtained. A test on this parameter can be carried out to discriminate among these nonnested, competitive models. Such an approach should be used in situa-

5. A Fréchet distribution with a tail index between 0 and -0.5 is compatible with both cases: for example, continuity would be obtained with a GARCH diffusion process as in Nelson (1990) and discontinuity with a jump process with the jump size drawn from a Student-*t* distribution.

tions where extremes matter. The loss of efficiency obtained when a small data subset (the extremes) is used may offset the bias when the whole data set (including central values without information relevant for the problem) is considered.

The empirical statistical results obtained in this article improve our understanding of extreme price movements which include booms and crashes. The results are statistical and complete the historical analysis by Kindleberger (1978) and economic studies by Schwert (1990*b*) on the volatile behavior around crashes, and by Jones, Sylla, and Wilson (1988) on the links between stock market crises and problems in the banking system. Here it is shown that even if these events may be explained by a variety of reasons (banking problems, news announcements, liquidity shocks), extreme values exhibit a regular statistical behavior as they are drawn from a well-known distribution. Such a result could be helpful in testing economic models of booms and crashes which could result from speculative bubbles, market structure deficiencies, or asymmetric information, as featured in Gennotte and Leland's (1990) model.

These results can be applied to problems in finance and economics where extreme values are significant. Two potential applications of extreme value theory are given below: margins in derivatives markets and minimum capital requirements for securities firms.

Margin setting in futures markets is well known to be sensitive to the occurrence of large price changes. Margin committees and brokers in futures markets face a trade-off when setting the margin level: a high level protects brokers against insolvent customers and then reinforces market integrity, but it also increases the cost supported by investors and in the end makes the market less attractive. Extreme value theory can be used to derive the margin level for a given probability of margin violation desired by margin committees or brokers. Longin (1995) proposes a new method to set margins along this line. The method takes into account the appropriate amount of extremes in the distribution of price changes and provides a simple analytical formula to compute the margin level.

A similar use of extreme value theory can be made in situations where risk is associated with the tails of the distribution rather than with the distribution as a whole. Regulators concerned with capital requirements for securities firms should be interested in the possibility of bankruptcy which could result from a extremely large change in the value of a firm's portfolio. Regulation currently imposed in the United States or that suggested by the Basles Committee on Banking supervision, does not however recognize this relation (Dimson and Marsh 1995). In this case risk measured by extreme value statistics may be more efficient than the usual measure of variance.

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