American Option Pricing Using A Markov Chain Approximation

Jin-Chuan Duan

Hong Kong University of Science and Technology

Correspondence to:
Prof. Jin-Chuan Duan
Department of Finance
Hong Kong University of Science & Technology
Clear Water Bay, Kowloon, Hong Kong
Tel: (852) 2358 7671; Fax: (852) 2358 1749
E-mail: jcduan@ust.hk
Web: http://www.bm.ust.hk/~jcduan

A two-way classification of option valuation problems

- Path-independent vs. path-dependent payoff functions

Path-independent:	European options, American options, Bermudan options, digital options
Path-dependent:	lookback options, Asian options, knock-out (in) options

- Path-independent vs. path-dependent underlying stochastic processes

Path-independent:	Black-Scholes model, CEV model, jump-diffusion
Path-dependent:	GARCH model

- In a numerical sense, it is typically easier to deal with path-dependence (a non-Markovian feature) arising from the payoff function than path-dependence inherent in the underlying dynamic.

- The traditional numerical methods are poor in either handling early exercise or the cases involving path-dependent payoffs (and worse for the cases involving path-dependent underlying dynamic).

Different numerical methods for option valuation

- Monte Carlo (quasi and pseudo) methods

  Exceedingly flexible in dealing with path-dependent payoffs and path-dependent underlying dynamic. Poor in handling options with early exercise possibilities. Computing time intensive even with some variance reduction technique.

- Finite difference/element methods

  Good at solving the option pricing problem that can be cast as a partial differential equation. Cannot deal with the discrete time valuation models.

- Lattice methods

  Hard to ensure recombination and harder still to accommodate path-dependent payoffs and/or underlying dynamics.

- Analytical approximation methods

  Highly valuation problem specific.

- Neural network methods

  Don't stand alone and are intended to be a complementary speed accelerator for real-time runs (see Hanke, 1997).

- Markov chain method (Duan & Simonato, 1999)

Black-Scholes Model

- **Asset return under the risk neutral probability measure :**

$$d\ln S_t = \left(r - \frac{1}{2}\sigma^2 \right) dt + \sigma dW_t$$

where

σ : volatility rate
r : risk free rate
W_t : standard Brownian motion

- **American option price with discrete exercise points :**

$$V(S_t, t) = \max\left\{ f(S_t, K), e^{-r} E^Q\left[V(S_{t+1}, t+1) \mid \Im_t \right] \right\}$$

where

$V(S_t, t)$: American option's price
K : Strike price
$f(S_t, K)$: European option's payoff
$V(S_T, T) = f(S_T, K)$

- **Discretize the underlying asset price**

Let $p_t \equiv -(r - \frac{1}{2}\sigma^2)t + \ln(S_t)$

Then,

$$dp_t = -(r - \frac{1}{2}\sigma^2)dt + d\ln(S_t)$$
$$= \sigma dW_t$$

- **Jarrow and Rudd's (1983, eq (13-18)) *n*-step binomial tree approximation**

Approximation target: p_t

Up and down moves: $u = \sigma\sqrt{\frac{T}{n}}$ and $d = -\sigma\sqrt{\frac{T}{n}}$

Probability: $q = \frac{1}{2}$

An n-step binomial tree has m discrete prices where $m = 2n + 1$.

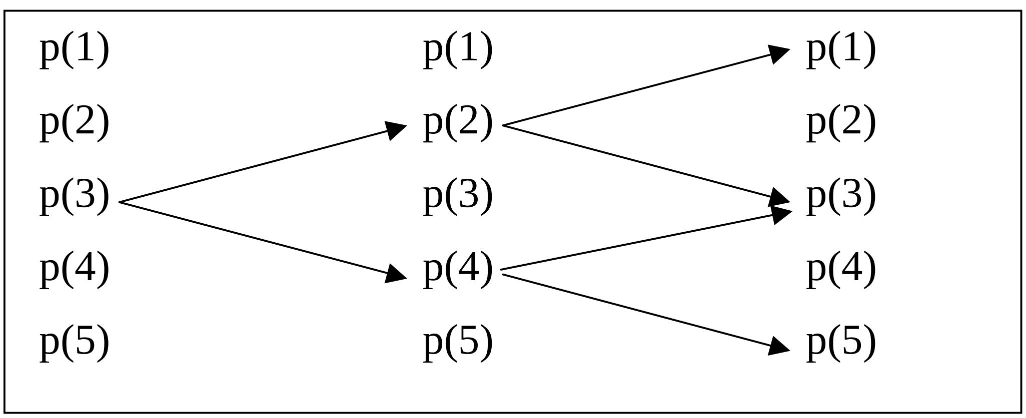
Vector for the logarithmic adjusted stock price:

$$\overline{P} = \left[p(1), p(2), \cdots, p(m) \right]$$

where

$$\begin{aligned} p(1) &= p_0 + nd \\ p(2) &= p_0 + (n-1)d \\ &\vdots \\ p(n+1) &= p_0 \\ p(n+2) &= p_0 + u \\ &\vdots \\ p(m) &= p_0 + nu \end{aligned}$$

Example: A 2-step binomial tree (m = 5)



A Markov chain interpretation with the following transition probability matrix

$$\Pi = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0.5 & 0 & 0.5 & 0 & 0 \\ 0 & 0.5 & 0 & 0.5 & 0 \\ 0 & 0 & 0.5 & 0 & 0.5 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

American option valuation numerically

$$\overline{V}(\overline{P},t) = \max\left\{g(\overline{P},K,t)\ ,\ e^{-r}\Pi\overline{V}(\overline{P},t+1)\right\}$$

where

$\overline{V}(\overline{P},t)$: American option's price
K : strike price
$g(\overline{P},K,t)$: European option's payoff
$\overline{V}(\overline{P},T) = g(\overline{P},K,T)$

Example: a put option

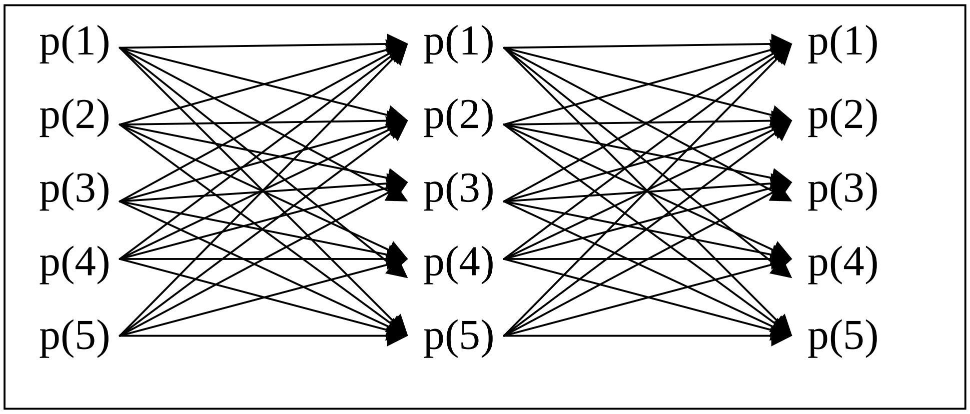
$$g(\overline{P},K,t) = \max\left\{k1 - \exp[(r - \frac{1}{2}\sigma^2)t1 + \overline{P}],0\right\}$$

Shortcoming of the lattice approach

1. Rigidity of geometry, i.e., the number of states is tied to the number of steps.
2. Recombined lattices are hard to construct.

- **A general Markov chain method**

The structure



Transition probability matrix

$$\Pi = \begin{bmatrix} \pi(1,1;\tau) & \pi(1,2;\tau) & \pi(1,3;\tau) & \pi(1,4;\tau) & \pi(1,5;\tau) \\ \pi(2,1;\tau) & \pi(2,2;\tau) & \pi(2,3;\tau) & \pi(2,4;\tau) & \pi(2,5;\tau) \\ \pi(3,1;\tau) & \pi(3,2;\tau) & \pi(3,3;\tau) & \pi(3,4;\tau) & \pi(3,5;\tau) \\ \pi(4,1;\tau) & \pi(4,2;\tau) & \pi(4,3;\tau) & \pi(4,4;\tau) & \pi(4,5;\tau) \\ \pi(5,1;\tau) & \pi(5,2;\tau) & \pi(5,3;\tau) & \pi(5,4;\tau) & \pi(5,5;\tau) \end{bmatrix}$$

where

$$\begin{aligned} \pi(i,j;\tau) &= \Pr^Q\left\{c(j) \le p_{t+\tau} < c(j+1) \mid p_t = \bar{p}(i)\right\} \\ &= \Pr^Q\left\{c(j) \le \quad p_t + \sigma\sqrt{\tau}\varepsilon_t < c(j+1) \mid p_t = \bar{p}(i)\right\} \\ &= \Pr^Q\left\{\frac{c(j) - \bar{p}(i)}{\sigma\sqrt{\tau}} \le \varepsilon_t < \frac{c(j+1) - \bar{p}(i)}{\sigma\sqrt{\tau}}\right\} \\ &= \Phi\left(\frac{c(j+1) - \bar{p}(i)}{\sigma\sqrt{\tau}}\right) - \Phi\left(\frac{c(j) - \bar{p}(i)}{\sigma\sqrt{\tau}}\right) \end{aligned}$$

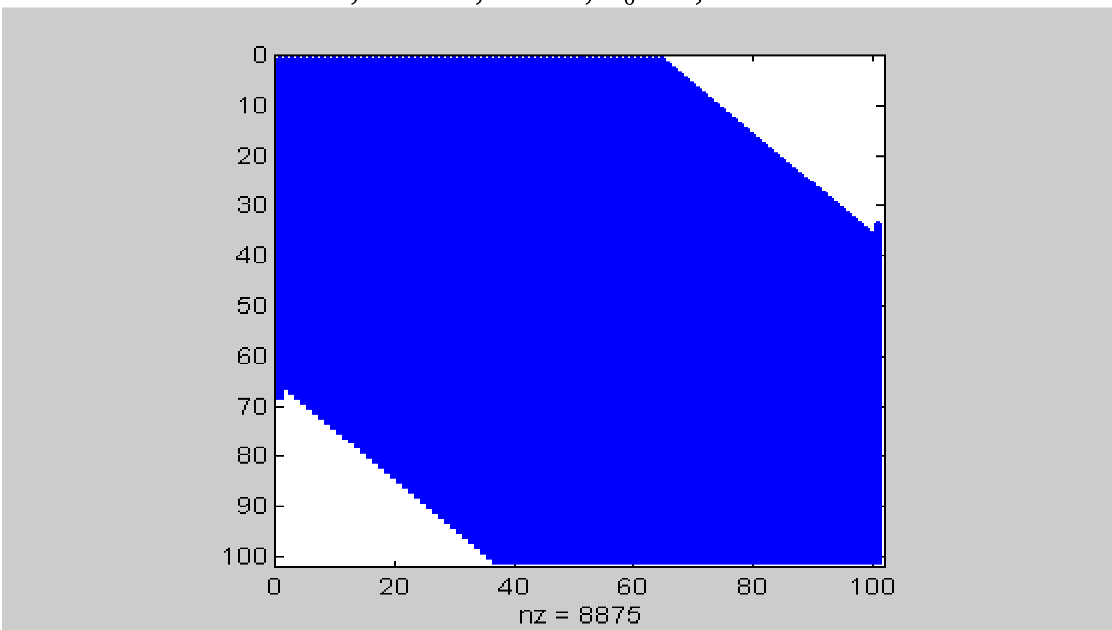
and $\Phi(.)$ is the standard normal distribution function

Density of the transition probability matrix

Fact: For a fixed partition, the density of the transition probability matrix depends on the length of one operation time interval

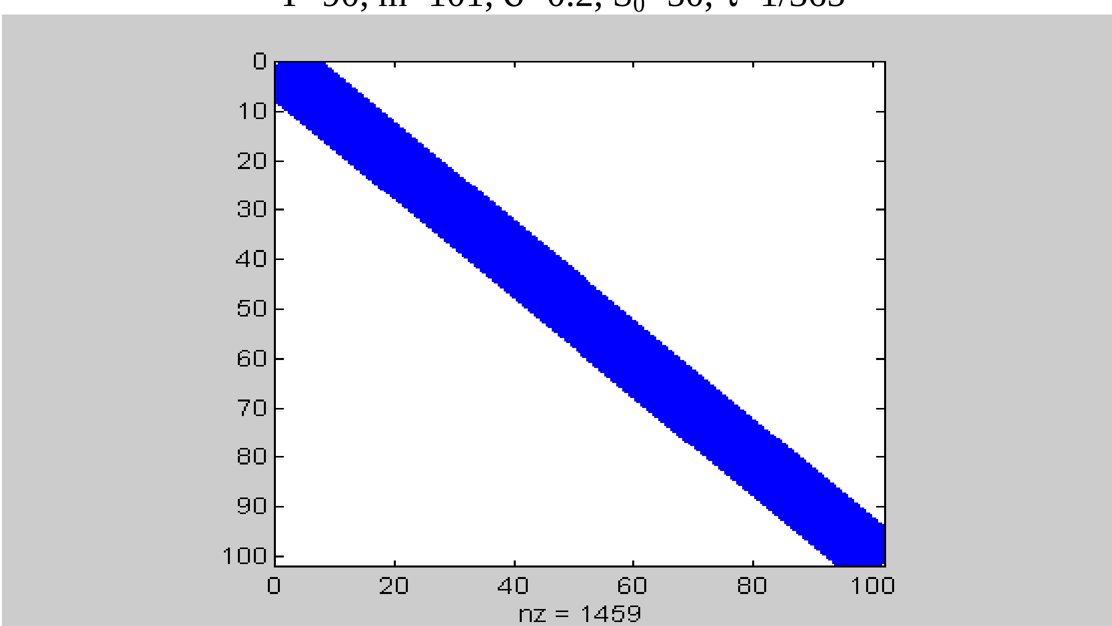
<u>Transition probability matrix using one step</u>

T=90, m=101, σ=0.2, S_0=50, τ=90/365



<u>Transition probability matrix using 90 steps</u>

T=90, m=101, σ=0.2, S_0=50, τ=1/365



• A numerical example

European put options in the Black and Scholes framework

Parameters:			
Maturity	1		
Strike Price	48		
Stock Price	50		
Interest Rate	0.05	d1	0.55411
Standard Deviation	0.2	d2	0.35411

Price computed by Black and Scholes Formula: 2.024003

Binomial Tree Approach

# of steps	2
Step size	0.141421

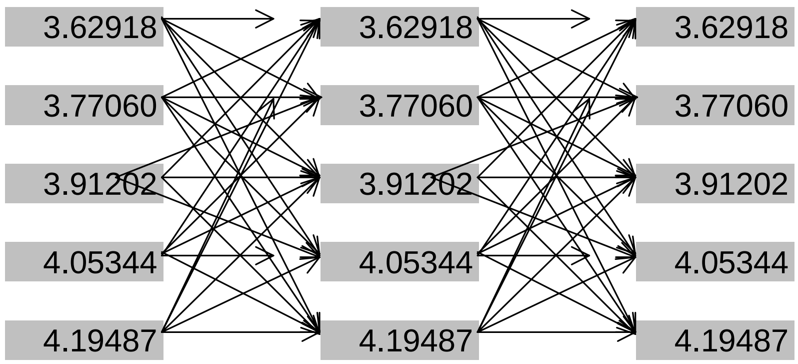
3.62918	3.62918	3.62918
3.77060	3.77060	3.77060
3.91202	3.91202	3.91202
4.05344	4.05344	4.05344
4.19487	4.19487	4.19487

Backstep 2	Backstep 1	Payoff	K - S
8.72325	8.94408	9.17050	9.17050
5.13971	4.47204	3.27191	3.27191
2.18081	1.59556	0.00000	-3.52273
0.77808	0.00000	0.00000	-11.34954
0.00000	0.00000	0.00000	-20.36532

Transition probability matrix

1	0	0	0	0
0.5	0	0.5	0	0
0	0.5	0	0.5	0
0	0	0.5	0	0.5
0	0	0	0	1

Markov Chain Approach # of prices 5



Backstep 2	Backstep 1	Payoff	K - S
5.71204	6.95589	9.17050	9.17050
3.91773	3.98155	3.27191	3.27191
1.96324	1.36892	0.00000	-3.52273
0.69951	0.24891	0.00000	-11.34954
0.17864	0.02115	0.00000	-20.36532

Transition Probability Matrix

0.69146	0.24173	0.06060	0.00598	0.00023
0.30854	0.38292	0.24173	0.06060	0.00621
0.06681	0.24173	0.38292	0.24173	0.06681
0.00621	0.06060	0.24173	0.38292	0.30854
0.00023	0.00598	0.06060	0.24173	0.69146

- **Numerical results for the Black-Scholes model** (Duan & Simonato, 1999)

$\Rightarrow$ Table 1 : European puts

$\Rightarrow$ Table 2 : American puts

GARCH Option Pricing Model

- **The reasons for GARCH option pricing**

  A. Option prices

  1. Volatility smile/smirk
  2. Term structure of volatilities
  3. Black-Scholes implied volatilities are higher than historical (or realized) volatilities

  B. Historical returns

  1. Volatility clustering
  2. Negative return skewness
  3. Excess return kurtosis

- **Asset return under the data generating probability measure *P***

$$\ln \frac{S_{t+1}}{S_t} = r + \lambda\sqrt{h_{t+1}} - \frac{1}{2}h_{t+1} + \sqrt{h_{t+1}}\xi_{t+1}$$

$$h_{t+2} = \beta_0 + \beta_1 h_{t+1} + \beta_2 h_{t+1}(\xi_{t+1} - \theta)^2$$

where

r : risk-free rate

λ : risk premium parameter

θ : leverage parameter

$\xi_{t+1} \sim N(0,1)$ with respect to P

- **Asset return under the risk neutral probability measure *Q* (Duan 1995)**

$$\ln \frac{S_{t+1}}{S_t} = r - \frac{1}{2} h_{t+1} + \sqrt{h_{t+1}} \varepsilon_{t+1}$$

$$h_{t+2} = \beta_0 + \beta_1 h_{t+1} + \beta_2 h_{t+1} (\varepsilon_{t+1} - \theta - \lambda)^2$$

where

$\varepsilon_{t+1} \sim N(0,1)$ with respect to Q

- **American option valuation in the GARCH framework**

$$V(S_t, h_{t+1}, t) = \max \left\{ f(S_t, K) , e^{-r} E^Q \left[V(S_{t+1}, h_{t+2}, t+1) | \Im_t \right] \right\}$$

where

$V(S_t, h_{t+1}, t)$: American option's price
K : Strike price
$f(S_t, K)$: European option's payoff
$V(S_T, h_{T+1}, T) = f(S_T, K)$

- **Discretize the underlying asset prices**

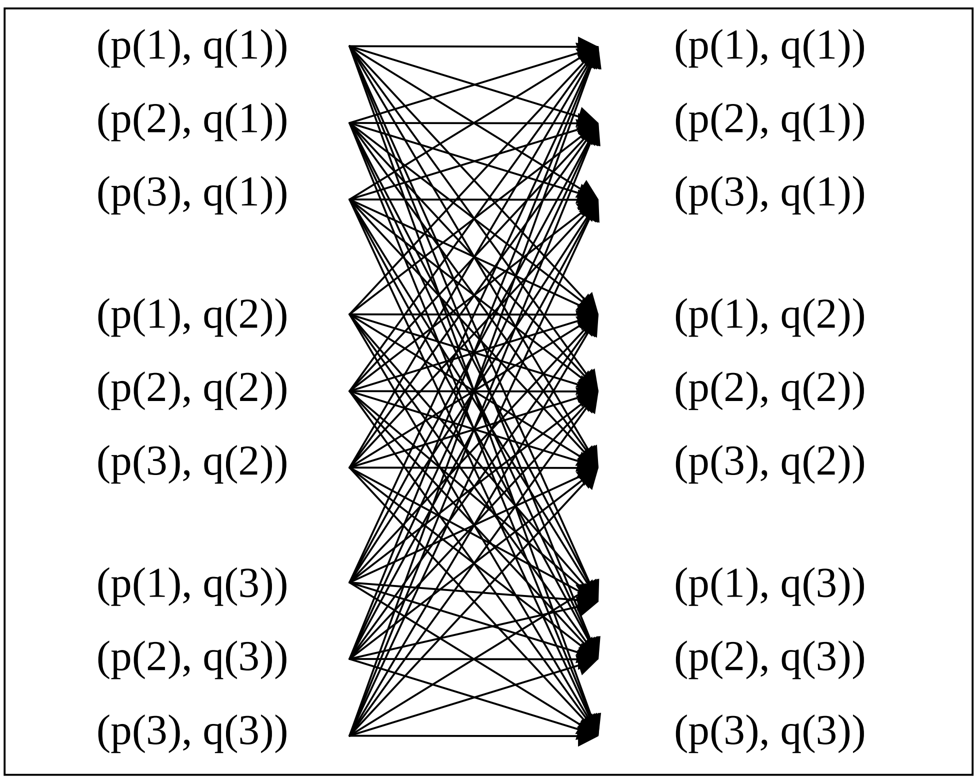
$$p_t = -(r - \frac{1}{2}h^*)t + \ln(S_t)$$

$$q_{t+1} = \ln(h_{t+1})$$

where

$$h^* = \frac{\beta_0}{1 - \beta_1 - \beta_2[1 + (\theta + \lambda)^2]}$$

Example: m=3, n=3



- **Transition probability matrix**

$$\Pi = \begin{bmatrix} \pi(1,1;1,1) & \pi(1,1;2,1) & \cdots & \pi(1,1;m,1) & \pi(1,1;1,2) & \cdots & \pi(1,1;m,n) \\ \pi(2,1;1,1) & \pi(2,1;2,1) & \cdots & \pi(2,1;m,1) & \pi(2,1;1,2) & \cdots & \pi(2,1;m,n) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \pi(m,1;1,1) & \pi(m,1;2,1) & \cdots & \pi(m,1;m,1) & \pi(m,1;1,2) & \cdots & \pi(1,2;m,n) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \end{bmatrix}$$

Since

$$\ln \frac{S_{t+1}}{S_t} = r - \frac{1}{2} h_{t+1} + \sqrt{h_{t+1}} \varepsilon_{t+1}$$

$$h_{t+2} = \beta_0 + \beta_1 h_{t+1} + \beta_2 h_{t+1} (\varepsilon_{t+1} - \theta - \lambda)^2$$

q_{t+2} is a deterministic function of q_{t+1}, p_{t+1}, p_t;
that is, $q_{t+2} = \Phi(q_{t+1}, p_{t+1}, p_t)$

$$\pi(i, j; k, l) =$$

$$\begin{cases} \Pr^Q \{ p_{t+1} \in C(k) \mid p_t = p(i), q_{t+1} = q(j) \} & \text{if } \Phi\big(q(j), p(k), p(i)\big) \in D(l) \\ 0 & \text{otherwise} \end{cases}$$

- **American option prices in the GARCH framework**

Price vector

$$\overline{P}' = \begin{bmatrix} p(1) & p(2) & \cdots & p(m) & \cdots & p(1) & p(2) & \cdots & p(m) \end{bmatrix}$$

American option price computation

$$\overline{V}(\overline{P},t) = \max\left\{ g(\overline{P},K,t), e^{-r}\Pi\overline{V}(\overline{P},t+1) \right\}$$

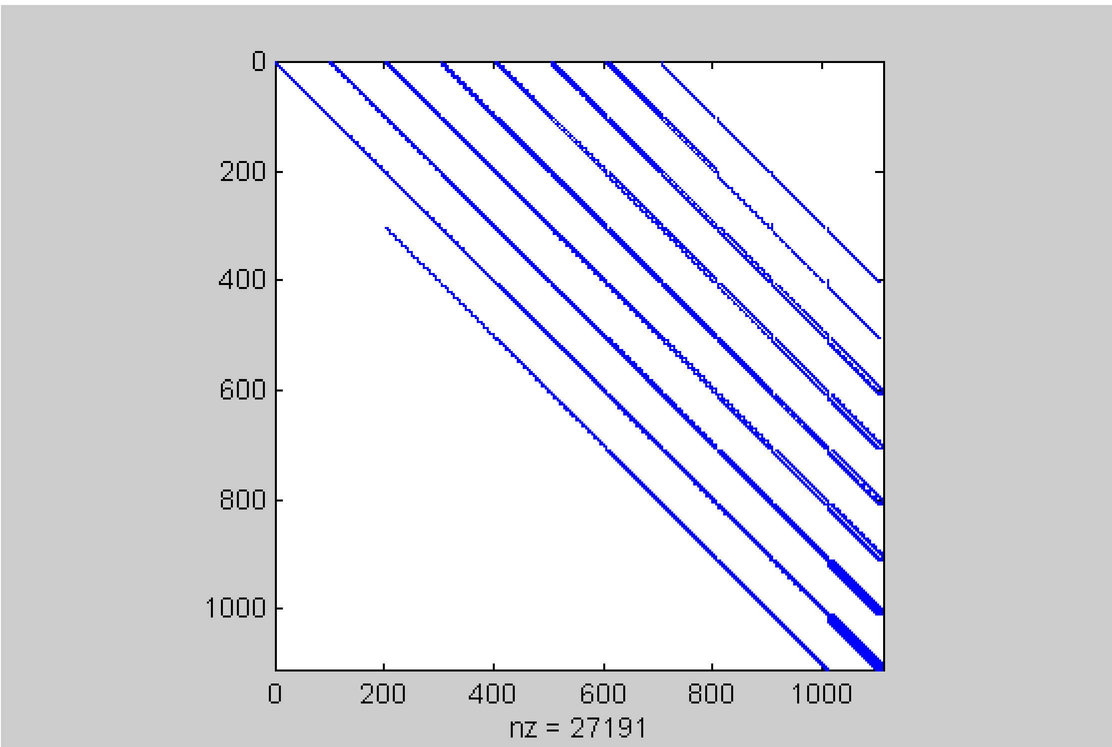
where

$\overline{V}(\overline{P},t)$: American option's price
K : Strike price
$g(\overline{P},K,t)$: European option's payoff function
$\overline{V}(\overline{P},T) = g(\overline{P},K,T)$

- **Density of the transition probability matrix (GARCH)**

m=101, *n*=11,

$$S_0 = 50, r = 0.05, \beta_0 = 0.00001, \beta_1 = 0.8, \beta_2 = 0.1, \lambda = 0.2$$



- **Numerical results for the GARCH model** (Duan & Simonato, 1999)

$\Rightarrow$ Table 3 : European put

$\Rightarrow$ Table 4 : American put

References

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